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Nonlocally Regularized Antibracket–Antifield Formalism and Anomalies in Chiral W_3 Gravity

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Abstract

The nonlocal regularization method, recently proposed in ref. [1, 2, 3], is extended to general gauge theories by reformulating it along the ideas of the antibracket-antifield formalism. From the interplay of both frameworks a fully regularized version of the field-antifield (FA) formalism arises, being able to deal with higher order loop corrections and to describe higher order loop contributions to the BRST anomaly. The quantum master equation, considered in the FA framework as the quantity parametrizing BRST anomalies, is argued to be incomplete at two and higher order loops and conjectured to reproduce only the one-loop corrections to the $\hbar^p$ anomaly generated by the addition of $O(\hbar^k)$, $k < p$, counterterms.

Chiral W_3 gravity is used to exemplify the nonlocally regularized FA formalism. First, the regularized one-loop quantum master equation is used to compute the complete one-loop anomaly. Its two-loop order, however, is shown to reproduce only the modification to the two-loop anomaly produced by the addition of a suitable one-loop counterterm, thereby providing an explicit verification of the previous statement for $p = 2$. The well-known universal two-loop anomaly, instead, is alternatively obtained from the BRST variation of the nonlocally regulated effective action. Incompleteness of the quantum master equation is thus concluded to be a consequence of a naive derivation of the FA BRST Ward identity.

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1 Introduction

Certain aspects of gauge theories, both at classical and at quantum level, seem presently most suitable treated in terms of the so-called antibracket-antifield formalism [4] (see [5] for recent reviews). As currently formulated, this proposal relies on BRST invariance as fundamental principle and the use of sources to deal with BRST transformations [6] [7]. In this way, when quantum aspects are under consideration, the field-antifield formalism (in short, FA) resembles the BRST approach since, in fact, the sources for BRST transformations of the latter are nothing but the antifields of the former. The Batalin-Vilkovisky approach, however, not only encompasses these previous ideas based on BRST invariance for quantizing gauge theories but also extends and generalizes them to more complicated situations (open algebras, reducible systems, etc.).

At the classical level, the FA formalism gives a general recipe to construct, out of a classical gauge action $S_0(\phi)$ and its gauge structure, a gauge-fixed action $\mathcal{S}(\Phi^A)$, suitable for path integral quantization, its BRST symmetry, $\delta\Phi^A = R^A(\Phi)$, and the higher order structure functions, $R^{A_n \ldots A_1}(\Phi)$, characterizing the underlying structure of the classical BRST symmetry. In practise, this is accomplished by first constructing from $S_0(\phi)$ and its gauge transformations, an action $S(z)$ in the so-called classical basis [8] of fields and antifields $z^a = \{\Phi^A, \Phi^*_A\}$, $A = 1, \ldots, N$, $a = 1, \ldots, 2N$, subject to the boundary conditions:

1. Classical limit: $S(\Phi, \Phi^*)|_{\Phi^*=0} = S_0(\phi)$,
2. Properness condition: $\text{rank}\,(S_{ab})|_{\text{on-shell}} = N$, with $S_{ab} \equiv \left(\frac{\partial_l \partial_r S}{\partial z^a \partial z^b}\right)$, and where on-shell means on the surface $\left\{\frac{\partial_r S}{\partial z^a} = 0\right\}$;

and satisfying the classical master equation

$$(S, S) = 0, \tag{1.1}$$

defined in terms of an odd symplectic structure, $(\cdot, \cdot)$, called antibracket

$$(X, Y) = \frac{\partial_r X}{\partial z^a} \zeta^{ab} \frac{\partial_l Y}{\partial z^b}\ , \qquad \text{where} \quad \zeta^{ab} \equiv (z^a, z^b) = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}. \tag{1.2}$$

Afterwards, by further performing a canonical transformation in the antibracket sense from the classical basis to the so-called gauge-fixed basis [4, 8], the BRST structure functions appear to be the coefficients in the antifield (or source) expansion of S

$$\begin{aligned} S(\Phi, \Phi^*) \quad &= \quad \mathcal{S}(\Phi) + \Phi^*_A R^A(\Phi) + \frac{1}{2} \Phi^*_A \Phi^*_B R^{BA}(\Phi) + \ldots \\ &\quad + \frac{1}{n!} \Phi^*_{A_1} \ldots \Phi^*_{A_n} R^{A_n \ldots A_1}(\Phi) + \ldots \end{aligned} \tag{1.3}$$

while the properness requirement translates in this basis to the condition

$$\text{rank}\,(\mathcal{S}_{AB})|_{\text{on-shell}} = N, \qquad \text{with} \qquad \mathcal{S}_{AB} \equiv \left(\frac{\partial_l \partial_r \mathcal{S}(\Phi)}{\partial \Phi^A \partial \Phi^B}\right),$$

i.e., propagators are well defined and the usual perturbation theory can be developped. Under such conditions, all the relations between the structure functions which characterize algebraically the classical BRST symmetry are completely encoded in the set of equations coming from (1.1), which may equivalently be called the classical BRST Ward identity.

Quantum corrections to this classical BRST symmetry and its underlying structure are most suitable analyzed in terms of the quantum counterpart of S (1.3), i.e., by considering the effective action $\Gamma(\Phi, \Phi^*)$ constructed, via the usual Legendre transformation with respect the sources J_A, from the generating functional

$$Z(J, \Phi^*) = \int \mathcal{D}\Phi \exp\left\{\frac{i}{\hbar}[W(\Phi, \Phi^*) + J_A \Phi^A]\right\}, \tag{1.4}$$

where the quantum action W

$$W = S + \sum_{p=1}^{\infty} \hbar^p M_p, \tag{1.5}$$

anticipates already the presence of local counterterms M_p guaranteeing finiteness of the theory while preserving (as far as possible) the BRST structure at quantum level. The quantum BRST structure and its possible breakdown appear then naturally described by the quantum analog of the classical BRST Ward identity (1.1)

$$\frac{1}{2}(\Gamma, \Gamma) = -i\hbar(\mathcal{A} \cdot \Gamma), \tag{1.6}$$

where the obstruction $(\mathcal{A} \cdot \Gamma)$ stands for the generating functional of the 1PI Green functions with one insertion of the composite field $\mathcal{A}$. This composite field $\mathcal{A}$ parametrizes thus potential departures from the classical BRST structure due to quantum corrections and is interpreted as the BRST anomaly.

The standard FA description provides for the anomaly $\mathcal{A}$ the expression

$$\mathcal{A} \equiv \left[\Delta W + \frac{i}{2\hbar}(W, W)\right](\Phi, \Phi^*), \tag{1.7}$$

with the operator Δ defined by

$$\Delta \equiv (-1)^{(A+1)} \frac{\partial_r}{\partial \Phi^A} \frac{\partial_r}{\partial \Phi^*_A}, \tag{1.8}$$

whereas its $\hbar$ expansion, $\mathcal{A} = \sum_{p=0} \hbar^{p-1} \mathcal{A}_p$, yields the form of the p-loop BRST anomalies

$$\begin{aligned}
\mathcal{A}_0 &= 1/2(S, S) \equiv 0, \\
\mathcal{A}_1 &= \Delta S + i(M_1, S), & (1.9) \\
\mathcal{A}_p &= \Delta M_{p-1} + \frac{i}{2}\sum_{q=1}^{p-1}(M_q, M_{p-q}) + i(M_p, S), \qquad p \geq 2. & (1.10)
\end{aligned}$$

Quantum BRST invariance in this framework is then claimed to be acquired if the anomaly $\mathcal{A}$ (1.7) in (1.6) vanishes, i.e., upon fulfillment through a local object W of the quantum master equation [4]

$$\left[\Delta W + \frac{i}{2\hbar}(W, W)\right](\Phi, \Phi^*) = 0, \tag{1.11}$$

which encodes at once the classical master equation (1.1) plus the set of recurrent equations for the counterterms M_p obtained by imposing the vanishing of (1.9) and (1.10).

The FA quantization program as it stands, however, presents some drawbacks. First, all the required manipulations are somewhat formal due to the ill-defined character of the path integral (1.4) unless an explicit regularization procedure is considered. At the level of the BRST Ward identity (1.6) this fact manifests itself in the ill-definiteness of the quantum master equation (1.11), due to the presence of the operator Δ (1.8) which generates $\delta(0)$ type singularities when

acting on local functionals as S or M_p. This problematics was first stressed in refs. [9, 10], where a Pauli-Villars (PV) regularization scheme was introduced to deal with the quantum master equation (1.11) and its relationship with the existence of anomalies was elucidated[1]. The main limitation of this PV regularized version of FA, however, is its inability to properly regularize two and higher order loop diagrams, i.e, only first order corrections (one loop) can be clearly studied.

A second major problem comes from the expression proposed by FA for the quantities parametrizing two and higher order loop anomalies, namely (1.10): while expression (1.9) has been tested in many examples to give the correct one-loop anomaly [9, 8, 11], it seems clear that, at higher orders, eq. (1.10) can not yield the complete p-loop anomaly. Indeed, on general grounds two and higher loop anomalies get contributions both from diagrams constructed out of the original action S (1.3) and, eventually, from new diagrams coming from the finite counterterms M_p, added a posteriori if required by BRST invariance. The absence in (1.10) of non-trivial contributions coming from the original action S indicates then that by no means these relations can generate the complete higher loop anomalies. In fact, expression (1.10) can be conjectured to be, as an explicit calculation will further confirm for $p = 2$, the one-loop correction generated by the counterterms M_k, $k < p$, to the p-loop anomaly. In view of that, the quantum master equation (1.11) in its present form can only be considered at most as the "one-loop form" of a more general expression, possibly defined in terms of suitable quantum generalizations of the operator Δ (1.8), $\Delta_q = \Delta + \hbar\Delta_1 + \ldots$, and of the antibracket (1.2) involved in its definition, whose closed form is still not known. This indicates that fundamental pieces, the ones reproducing precisely two and higher order loop anomalies, are missed in standard FA derivations of the quantum master equation (1.11), so that the proposed FA form (1.7) for the BRST anomaly is incomplete at two and higher loops. It should be stressed, however, that this incompleteness by no means questions the generic form of the BRST Ward identity (1.6) –ensured, as pointed out for instance in [12], by the celebrated Lam's action principle [13]– but *only the specific form* (1.7) *of the anomaly insertion proposed in the standard FA formalism and its formal derivation.*

Confrontation with these drawbacks, thus, leads to looking for a regularized version of the FA formalism being able to deal with higher order loop corrections and to describe the correct structure of the higher order contributions to the "one-loop" BRST anomaly (1.7). In this paper, we pursue the first part of this program by reconsidering the ideas of a new regularization method, the so-called nonlocal regularization, recently introduced in [1, 2, 3]. Basically, this method –originally applied to irreducible theories with closed gauge algebra– consists in modifying the original BRST invariant action by including nonlocal, higher order interactions depending on a given cut-off Λ^2 (nonlocalization), in such a way that the resulting theory is finite at any order. The regulated theory results then to be invariant at the classical (tree) level under a nonlocal, distorted version of the original BRST symmetry. Quantum BRST invariance, in turn, will require in general further introduction of suitable measure factors, i.e., the "nonlocal" analogs of the counterterms M_p, which should also be regularized along similar lines.

Incorporation of all these ideas in the FA framework appears then to be natural and fruitful since, while giving the way to extend the formalism of refs. [1, 2, 3] to general gauge theories (open algebras, reducible systems, etc), it also provides as a result a fully regularized version of the antibracket-antifield formalism, suitable for the study of the BRST Ward identity (1.6) proposed in this framework and related anomaly issues. In the end, the resulting nonlocally regularized FA formalism is seen to properly deal with higher order loop corrections and anomalies when computing them directly from the BRST variation of the effective action Γ –as the considered example, W_3, shows. However, and as stated before, when approached by using the form (1.7) of the anomaly, it only reproduces correctly one-loop anomalies and one-loop corrections to

[1]For an updated approach to this subject, see ref. [11].

higher loop anomalies coming from counterterms, providing thus an explicit verification of the previous assertion. The analysis of the complete form of the anomaly and of the quantum master equation, however, lies outside the scope of this paper and its study is left as an open problem.

The paper has been organized as follows. In section 2, the nonlocal regularization method of refs.[2, 3] is briefly reviewed in order to set out notation and to slightly improve it to describe in generality systems with either first or second order kinetic term in space-time derivatives. Section 3 deals with the extension of the previous method to general gauge theories by reformulating it according to the ideas of the antibracket-antifield formalism, both at classical and at quantum level. In particular, a prescription is given to obtain the regularized form of the proper solution of the classical master equation S and of its quantum extension W and a regularized version of the quantum master equation is provided after analyzing the action of Δ on this regularized quantum action. The theoretical framework is then exemplified in section 4 by applying it to chiral W_3 gravity. First of all, the one and two-loop orders of the regularized quantum master equation are used to compute the complete one-loop anomaly and the correction to the universal two-loop anomaly produced by the addition of a one-loop counterterm, respectively, while the universal two-loop anomaly is afterwards evaluated from the BRST variation of the nonlocally regulated effective action. Section 5 summarizes the conclusions and indicates where the incompleteness of the quantum master equation can originate by analyzing its standard FA derivation. Finally, an appendix devoted to the computation of the general form of the functional traces involved in the calculations of section 4 is provided.

2 Nonlocal Regularization

The aim of nonlocal regularization is to provide a systematic, field theoretic formulation of the main ideas contained in Schwinger's proper time method [14]. In this approach, propagators are realized in terms of integrals over real parameters $t_i \in [0, \infty)$, $i = 1, \ldots, n$, so that ultraviolet divergences become, after integration over loop momenta, singularities around the region $t_i = 0$. Nonlocal regularization proposes to split the original divergent loop integrals as a sum over loop contributions coming from the original fields with modified propagators plus extra loop contributions generated by a set of auxiliary fields, in such a way that original singularities be contained in the loops composed solely of auxiliary fields. Elimination of these auxiliary fields by putting them on–shell gets rid of their quantum fluctuations, and so of its divergent loops, and regularizes the original theory.

In what follows, we briefly review the nonlocal regularization method along the lines of refs.[2, 3], with some slight modifications in order to describe in a unified way systems with either first or second order kinetic term in space-time derivatives. This summary shall also serve to fix our conventions and notations.

2.1 The Nonlocally Regulated Action

Consider a theory defined by a classical action $\mathcal{S}(\Phi)$ (as for instance, the gauge-fixed action in (1.3)), which depends on the set of fields Φ^A, $A = 1, \ldots, N$, with statistics $\epsilon(\Phi^A) \equiv A$. The so-called nonlocal regularization applies to theories which have a sensible perturbative expansion, i.e., to theories for which the action $\mathcal{S}(\Phi)$ admits a decomposition into free and interacting parts

$$\mathcal{S}(\Phi) = F(\Phi) + I(\Phi), \qquad \text{with} \qquad F(\Phi) = \frac{1}{2}\Phi^A \mathcal{F}_{AB} \Phi^B, \tag{2.1}$$

and where $I(\Phi)$ is assumed to be analytic in Φ^A around $\Phi^A = 0$.

Introduce now a cut–off or regulating parameter Λ^2 and a field independent operator T_{AB}, invertible but otherwise arbitrary. This operator is chosen such that from its inverse $(T^{-1})^{AB}$

and the kinetic operator $\mathcal{F}_{AB}$ in (2.1), a second order derivative "regulator" $\mathcal{R}^A_B$ arises through the combination

$$\mathcal{R}^A_B = (T^{-1})^{AC}\mathcal{F}_{CB}.$$

Afterwards, from this object, the smearing operator, ε^A_B, and the shadow kinetic operator, $\mathcal{O}^{-1}_{AB}$, are constructed

$$\varepsilon^A_B = \exp\left(\frac{\mathcal{R}^A_B}{2\Lambda^2}\right), \tag{2.2}$$

$$\mathcal{O}^{-1}_{AB} = T_{AC}(\tilde{\mathcal{O}}^{-1})^C_B = \left(\frac{\mathcal{F}}{\varepsilon^2 - 1}\right)_{AB}, \tag{2.3}$$

with $(\tilde{\mathcal{O}})^A_B$ defined as

$$\tilde{\mathcal{O}}^A_B = \left(\frac{\varepsilon^2 - 1}{\mathcal{R}}\right)^A_{\ B} = \int_0^1 \frac{\mathrm{d}t}{\Lambda^2}\exp\left(t\frac{\mathcal{R}^A_B}{\Lambda^2}\right).$$

Now, for each field Φ^A introduce an auxiliary field Ψ^A with the same statistics –the so-called shadow fields– and couple both sets of fields through the auxiliary action

$$\tilde{\mathcal{S}}(\Phi,\Psi) = F(\hat{\Phi}) - A(\Psi) + I(\Phi + \Psi), \tag{2.4}$$

with $A(\Psi)$, the kinetic term for the auxiliary fields, constructed with the help of (2.3) as

$$A(\Psi) = \frac{1}{2}\Psi^A\mathcal{O}^{-1}_{AB}\Psi^B,$$

and where the "smeared" fields $\hat{\Phi}^A$ appearing in the free part of the auxiliary action (2.4) are defined, using (2.2), by

$$\hat{\Phi}^A \equiv (\varepsilon^{-1})^A_B\Phi^B = \Phi^B(\varepsilon^{-1})_B^{\ A}, \tag{2.5}$$

with $(\varepsilon^{-1})_B^{\ A}$ the supertranspose of $(\varepsilon^{-1})^A_B$, i.e., $(\varepsilon^{-1})_B^{\ A} = (\varepsilon^{-1})^A_B(-1)^{B(A+B)}$.

From the above construction, it should be clear that the perturbative theory described by (2.4), when only external Φ lines are considered, and (2.1) are exactly the same. Indeed, for a given diagram, Φ external lines connect either to a smeared propagator

$$\left(\frac{i\varepsilon^2}{\mathcal{F} + i\epsilon}\right)^{AB} = -i\left[\int_1^\infty \frac{\mathrm{d}t}{\Lambda^2}\exp\left(t\frac{\mathcal{R}^A_C}{\Lambda^2}\right)\right](T^{-1})^{CB}, \tag{2.6}$$

or to a shadow propagator

$$-i\mathcal{O}^{AB} = \left(\frac{i(1-\varepsilon^2)}{\mathcal{F}}\right)^{AB} = -i\left[\int_0^1 \frac{\mathrm{d}t}{\Lambda^2}\exp\left(t\frac{\mathcal{R}^A_C}{\Lambda^2}\right)\right](T^{-1})^{CB}, \tag{2.7}$$

which diagramatically are going to be represented by "unbarred" and "barred" lines, respectively, (Fig. 1)

Fig. 1. Unbarred and barred propagators.

On the other hand, the specific form of the interaction in (2.4), $I(\Phi+\Psi)$, associates, to a given original diagram with n internal lines, 2^n diagrams consisting in all possible combinations formed with both smeared (2.6) and shadow (2.7) internal propagators. Then, the sum of (2.6) and (2.7) being the original Φ propagator, it is concluded that the sum of all these 2^n contributions yields the original diagram with n original propagators in its internal lines. However, the special form of the shadow propagator (2.7) as an integral over t in the region $[0,1]$ leads the contribution composed solely from "barred" lines to isolate the divergent part of the original diagram, i.e., the one coming from the integration over the hypercube $[0,1]^n$ in parameter space. Cancellation of this diagram contribution and, in the general case, of the quantum fluctuations of the shadow fields, regularizes thus the theory.

The process of eliminating quantum fluctuations associated uniquely with shadow fields, or conversely, of the closed loops formed solely with barred lines by hand, is equivalently implemented at the path integral level by putting the auxiliary fields Ψ on-shell. In this second strategy, the classical shadow field equations of motion

$$\frac{\partial_r \tilde{\mathcal{S}}(\Phi,\Psi)}{\partial \Psi^A}=0 \quad \Rightarrow \quad \Psi^A=\left(\frac{\partial_r I}{\partial \Phi^B}(\Phi+\Psi)\right)\mathcal{O}^{BA}, \tag{2.8}$$

should be solved, in general, in a perturbative fashion and its classical solution $\bar{\Psi}_0(\Phi)$ substituted in the auxiliary action (2.4). The result of this process is the nonlocalized action to be used in regularized computations

$$\mathcal{S}_\Lambda(\Phi)\equiv \tilde{\mathcal{S}}(\Phi,\bar{\Psi}_0(\Phi)). \tag{2.9}$$

The expansion of $\mathcal{S}_\Lambda(\Phi)$ in $\bar{\Psi}_0$ can then be seen to generate the smeared kinetic term $F(\hat{\Phi})$, the original interaction $I(\Phi)$ plus an infinite series of new nonlocal interaction terms, whose effect is completely equivalent to that of the mixed loops of unbarred and barred lines in the previous formulation, and which arise from them after putting the shadow fields on-shell. All these new nonlocal interaction terms are of $O(\frac{1}{\Lambda^2})$ and, therefore, vanishing in the limit $\Lambda^2\to\infty$, so that $\mathcal{S}_\Lambda(\Phi)\to\mathcal{S}(\Phi)$ in this limit and the original theory is recovered. This result can also be obtained by directly considering in $\mathcal{S}_\Lambda(\Phi)$ the following limits when $\Lambda^2\to\infty$:

$$\varepsilon\to 1, \qquad \mathcal{O}\to 0, \qquad \bar{\Psi}_0(\Phi)\to 0. \tag{2.10}$$

In conclusion, the nonlocally regulated theory can thus be realized in two ways: either by using the auxiliary action (2.4) and eliminating by hand closed loops formed solely with barred lines, or by putting the fields Ψ on–shell. The first strategy is the best when performing diagramatic calculations, since the Feynman rules which correspond to the auxiliary action (2.4) are essentially as simple as those of the original local theory. Use of the nonlocally regulated action (2.9), instead, is more convenient when doing algebraic manipulations, as the ones involved in the analysis of the regulated version of the quantum master equation (1.11). Examples of both procedures will be explicitly considered in the specific model of chiral W_3 gravity.

2.2 Nonlocally Regulated Symmetries

Any local quantum field theory can be made ultraviolet finite just by introducing the smearing operator (2.2) and by working with the modified action $F(\hat{\Phi})+I(\Phi)$, $\hat{\Phi}$ being the smeared field (2.5) [1, 2, 3][2]. However, this form of nonlocalization generally spoils any sort of gauge symmetry or its associated BRST symmetry, leading to the breakdown of the corresponding Ward identities already at the tree level and threatening all the benefits derived by its use in the study of perturbative unitarity and renormalizability issues.

[2] See also [15] for an earlier approach and [16] for an alternative formulation of this idea in the context of chiral perturbation theory.

The form of nonlocalization presented above, instead, has the merit of preserving at tree level a distorted version of any of the original continuous symmetries of the theory, and in particular, of the BRST (or of the original gauge) symmetry. Indeed, assume that the original action (2.1) is invariant under the infinitesimal transformation

$$\delta\Phi^A = R^A(\Phi). \tag{2.11}$$

Then, the auxiliary action results to be invariant under the auxiliary infinitesimal transformations

$$\tilde{\delta}\Phi^A = (\varepsilon^2)^A_B R^B(\Phi+\Psi), \qquad \tilde{\delta}\Psi^A = (1-\varepsilon^2)^A_B R^B(\Phi+\Psi), \tag{2.12}$$

while the nonlocally regulated action $\mathcal{S}_\Lambda(\Phi)$ (2.9) becomes invariant under

$$\delta_\Lambda\Phi^A = (\varepsilon^2)^A_B R^B(\Phi+\bar{\Psi}_0(\Phi)), \tag{2.13}$$

with $\bar{\Psi}_0(\Phi)$ the classical solution of (2.8). The proof of these statements is straightforward and can be found in the original reference [2], to which the reader is referred for further details.

The nonlocalization procedure provides thus a definite way to regularize the continuous symmetries of a given theory. However, although apparently symmetries are maintained, its underlying structure is changed somehow. Indeed, it is possible to see [2] that, for instance, if the original symmetry (2.11) has a closed algebra, its corresponding nonlocal version δ_Λ (2.13) closes in general only on-shell. In other words, if, simbolically $[\delta_1,\delta_2]=\delta_3$, then in general

$$[\delta_{\Lambda,1},\delta_{\Lambda,2}]\Phi^A = \delta_{\Lambda,3}\Phi^A + \left[(\varepsilon^2)^A_B\,\Omega^{BC}_{12}(\Phi+\bar{\Psi}_0(\Phi))\,(\varepsilon^2)^D_C(-1)^D\right]\frac{\partial_r\mathcal{S}_\Lambda(\Phi)}{\partial\Phi^D}, \tag{2.14}$$

with Ω^{AB}_{12} given by

$$\Omega^{AB}_{12}(\Phi) = \left(\frac{\partial_r R^A_1}{\partial\Phi^C}\,K^{CD}\,\frac{\partial_l R^B_2}{\partial\Phi^D} - (1\leftrightarrow 2)\right)(\Phi),$$

and where the (inverse of the) operator K^{AB} is defined as

$$(K^{-1})_{AB}(\Phi) = \left[(\mathcal{O}^{-1})_{AB} - \frac{\partial_l\partial_r I(\Phi)}{\partial\Phi^A\partial\Phi^B}\right]. \tag{2.15}$$

This phenomenon is well-known to happen upon elimination of a given subset of (auxiliary) fields through their classical equations of motion [5, 17].

Continuing in this way, that is to say, by taking more and more commutators of nonlocally regulated symmetries (2.13), the higher order structure functions characterizing the original symmetry are seen to be modified in much the same direction by the nonlocalization process. In summary, the complete symmetry structure associated with the nonlocally regulated transformation (2.13) results to be a distorted version of the original one: not only the action and the transformations are distorted, but also the underlying structure functions characterizing the original symmetry are also regulated and changed. The nonlocal regularization procedure works therefore by distorting or regularizing the complete classical structure of the original theory, in such a way that in the limit $\Lambda^2\to\infty$ the regulated structure converges to the original one.

3 Nonlocally regularized Antibracket–Antifield formalism

In view of the previous results, the question naturally arises, for a given BRST quantized gauge theory, of the relationship between the original structure functions and its underlying algebraic BRST structure with their regulated versions. It is here where the antibracket-antifield

framework turns out to be useful in characterizing and determining the structure of the regulated BRST symmetry in terms of the original one. The results in ref. [2], mainly designed for irreducible theories with closed gauge algebra, partially answered the question by giving the explicit form of the regulated action (2.9) and of the BRST transformations (2.13). Using these forms as starting point, or as boundary conditions, and combining them with well-known FA results, the regulated BRST classical structure of a general gauge theory can be completely elucidated from the knowledge of the original one. As a byproduct of this nonlocalization process, and after extending it to the quantum action W, a nonlocally regularized antibracket-antifield formalism comes out in a natural way.

3.1 Nonlocalization of the proper solution

Consider a FA quantized gauge theory and assume that the gauge fixed action $\mathcal{S}(\Phi)$ in the proper solution of the classical master equation (1.3) admits a suitable perturbative decomposition of the form (2.1). From the general results stated in the introduction, it is concluded that the regulated version of the structure functions should come out as coefficients in the antifield expansion of a proper solution having as lowest order terms the nonlocal action (2.9) and its nonlocal BRST transformations (2.13). The uniqueness of these regulated structure functions should be understood, of course, modulo canonical transformations.

This nonlocal proper solution can be obtained by rewriting the process described in sect. 2 according to FA ideas. Enlarge thus first the original field-antifield space by introducing the shadow fields and their antifields $\{\Psi^A, \Psi^*_A\}$. Then, in this extended space, an auxiliary proper solution should be constructed as a preliminary step, which incorporates the auxiliary action $\tilde{\mathcal{S}}(\Phi,\Psi)$ (2.4), corresponding to the gauge-fixed action $\mathcal{S}(\Phi)$, its BRST symmetry (2.12) and the yet unknown associated higher order structure functions. In doing so, it is realized that the auxiliary BRST transformations (2.12), which should appear in the auxiliary proper solution as the first order term in the antifields

$$\left[\Phi^*_A(\varepsilon^2)^A_B + \Psi^*_A(1-\varepsilon^2)^A_B\right] R^B(\Phi+\Psi),$$

can be obtained from the term $\Phi^*_A R^A(\Phi)$ in the original proper solution through the replacements

$$\Phi^*_A \to \left[\Phi^*_B(\varepsilon^2)^B_A + \Psi^*_B(1-\varepsilon^2)^B_A\right] \equiv \Theta^*_A, \tag{3.1}$$

$$R^A(\Phi) \to R^A(\Phi+\Psi) \equiv R^A(\Theta). \tag{3.2}$$

It is therefore tempting to extend this rule to the higher order antifield terms in (1.3) in order to reconstruct the auxiliary higher order structure functions, i.e., to consider (3.1) and

$$R^{A_n\ldots A_1}(\Phi) \to R^{A_n\ldots A_1}(\Phi+\Psi) = R^{A_n\ldots A_1}(\Theta), \tag{3.3}$$

and write as an ansatz for the auxiliary proper solution

$$\begin{aligned}\tilde{S}(\Phi,\Phi^*;\Psi,\Psi^*) \quad &= \quad \tilde{\mathcal{S}}(\Phi,\Psi) + \Theta^*_A R^A(\Theta) + \frac{1}{2}\Theta^*_A\Theta^*_B R^{BA}(\Theta) + \ldots \\ &\quad + \frac{1}{n!}\Theta^*_{A_1}\ldots\Theta^*_{A_n} R^{A_n\ldots A_1}(\Theta) + \ldots .\end{aligned} \tag{3.4}$$

It is not difficult to check that (3.4) is indeed a proper solution of (1.1) in the extended space. One should realize, first of all, that the linear combinations $\{\Theta^A, \Theta^*_A\}$ defined in (3.1), (3.2), from which the antifield dependent part of (3.4) is constructed, are conjugated variables,

i.e., $(\Theta^A, \Theta^*_B) = \delta^A_B$. This fact suggests to use a new set of fields and antifields obtained by completing the subset $\{\Theta^A, \Theta^*_A\}$ with the linear combinations $\{\Sigma^A, \Sigma^*_A\}$ defined as

$$\Sigma^A = \left[(1-\varepsilon^2)^A_B \Phi^B - (\varepsilon^2)^A_B \Psi^B\right], \qquad \Sigma^*_A = \Phi^*_A - \Psi^*_A,$$

in such a way that the linear transformation

$$\{\Phi^A, \Phi^*_A; \Psi^A, \Psi^*_A\} \to \{\Theta^A, \Theta^*_A; \Sigma^A, \Sigma^*_A\}, \tag{3.5}$$

be canonical in the antibracket sense. In terms of this new set of variables the auxiliary action (3.4) acquires the form

$$\tilde{S}(\Theta, \Theta^*; \Sigma, \Sigma^*) = S(\Theta, \Theta^*) + \frac{1}{2}\Sigma^A \left[\frac{\mathcal{F}}{\varepsilon^2} + \frac{\mathcal{F}}{(1-\varepsilon^2)}\right]_{AB} \Sigma^B, \tag{3.6}$$

where $S(\Theta, \Theta^*)$ is the original proper solution (1.3) with arguments $\{\Theta^A, \Theta^*_A\}$. Therefore, since no antifields Σ^* are present and $S(\Theta, \Theta^*)$ verifies the classical master equation, the action (3.6) fulfills (1.1) as well. The canonical nature of the linear transformation (3.5) further ensures the fulfillment of (1.1) by the auxiliary action (3.4). On the other hand, properness of (3.6), derived from the properness of the original solution S and the fact that the kinetic term of the Σ^A fields has maximum rank, guarantees properness of the auxiliary action (3.4) due to the conservation of the proper character of a solution of (1.1) upon canonical transformations. It can also be checked by direct inspection of the auxiliary action (3.4). This completes the first part of the process, thus indicating that replacements (3.1), (3.2) and (3.3) provide an effective way to describe the structure functions characterizing the auxiliary BRST structure in terms of the original BRST structure, without performing any concrete computation.

The next step of the process, namely, nonlocalization, is acquired by eliminating the shadow fields and antifields by means of the standard elimination process of auxiliary fields in the FA framework[3], that is, by substituting the shadow fields by the solutions of their classical equations of motion, while putting their antifields to zero. In other words, the candidate to nonlocal proper solution is obtained from (3.4) as

$$S_\Lambda(\Phi, \Phi^*) = \tilde{S}(\Phi, \Phi^*; \bar{\Psi}, \Psi^* = 0), \tag{3.7}$$

with $\bar{\Psi} \equiv \bar{\Psi}(\Phi, \Phi^*)$ the solution of the classical equations of motion

$$\frac{\partial_r \tilde{S}(\Phi, \Phi^*; \Psi, \Psi^* = 0)}{\partial \Psi^A} = 0. \tag{3.8}$$

The substitution of the subset of fields $\{\Psi^A, \Psi^*_A\}$ in this way leads (3.7) to fulfill again (1.1). Indeed, a simple computation using relations

$$\frac{\partial_r S_\Lambda}{\partial \Phi^A} = \left.\frac{\partial_r \tilde{S}}{\partial \Phi^A}\right|, \qquad \frac{\partial_l S_\Lambda}{\partial \Phi^*_A} = \left.\frac{\partial_l \tilde{S}}{\partial \Phi^*_A}\right|, \tag{3.9}$$

where the above restriction means on the surface $\{\Psi = \bar{\Psi}(\Phi, \Phi^*), \Psi^* = 0\}$, and which hold as a consequence of (3.8), leads to

$$\frac{1}{2}(S_\Lambda, S_\Lambda) = \left.\frac{\partial_r \tilde{S}}{\partial \Phi^A}\frac{\partial_l \tilde{S}}{\partial \Phi^*_A}\right| = \left.\left(\frac{\partial_r \tilde{S}}{\partial \Phi^A}\frac{\partial_l \tilde{S}}{\partial \Phi^*_A} + \frac{\partial_r \tilde{S}}{\partial \Psi^A}\frac{\partial_l \tilde{S}}{\partial \Psi^*_A}\right)\right| = \left.\frac{1}{2}(\tilde{S}, \tilde{S})\right| = 0. \tag{3.10}$$

[3]For an study of the elimination process of auxiliary fields and antifields in the antibracket-antifield formalism and its consequences, the reader is referred to [5, 17].

On the other hand, in order to verify properness of (3.7), as well as to finally determine the nonlocally regulated version of the structure coefficients, it is worth to investigate in more detail the antifield expansion of S_Λ. To this end, the equations of motion (3.8) for the fields Ψ^A, which explicitly read

$$\Psi^A = \left[\frac{\partial_r I}{\partial \Phi^B}(\Phi+\Psi) + \Phi^*_C(\varepsilon^2)^C_D R^D_B(\Phi+\Psi) + \mathcal{O}((\Phi^*)^2)\right]\mathcal{O}^{BA}, \tag{3.11}$$

with $R^A_B = \frac{\partial_r R^A(\Phi)}{\partial \Phi^B}$, can be solved perturbatively in powers of antifields in the standard way. The lowest order of equation (3.11)

$$\Psi^A = \left(\frac{\partial_r I}{\partial \Phi^B}(\Phi+\Psi)\right)\mathcal{O}^{BA}, \tag{3.12}$$

results to be equation (2.8), as expected. Expanding around its solution $\bar\Psi_0$ at first order in the antifields, $\Psi^A_1(\Phi,\Phi^*) = \bar\Psi^A_0(\Phi) + \Phi^*_B D^{BA}(\Phi)$, and plugging this expression in (3.11), yields

$$\bar\Psi^A_1(\Phi,\Phi^*) = \bar\Psi^A_0(\Phi) + \Phi^*_B(\varepsilon^2)^B_C R^C_D K^{DA}(\Phi+\bar\Psi_0), \tag{3.13}$$

with (the inverse of) $K^{DA}(\Phi)$ given by (2.15). Applying recursively this procedure, an expression can be obtained for $\bar\Psi(\Phi,\Phi^*)$ at any desired order in antifields in terms of the antifield independent solution $\bar\Psi_0$ of eq. (3.12).

Solution (3.13) is enough to figure out the form of the regulated structure functions up to second order in the antifields and to compare the results with the ones presented in sect. 2. Indeed, recall the form of the nonlocal proper solution (3.7) in terms of the auxiliary proper solution (3.4). The lowest order term of (3.4) yields

$$\tilde{\mathcal{S}}(\Phi,\bar\Psi_1) = \tilde{\mathcal{S}}(\Phi,\bar\Psi_0) - \frac{1}{2}\left[\Phi^*_A\,(\varepsilon^2)^A_C\,R^C_D\,\Phi^*_B\,(\varepsilon^2)^B_F\,R^F_E\,K^{ED}\right](\Phi+\bar\Psi_0),$$

the first order term results in

$$\Phi^*_A(\varepsilon^2)^A_B R^B(\Phi+\bar\Psi_1) =$$
$$\Phi^*_A(\varepsilon^2)^A_B R^B(\Phi+\bar\Psi_0) + \left[\Phi^*_A\,(\varepsilon^2)^A_C\,R^C_D\,\Phi^*_B\,(\varepsilon^2)^B_F\,R^F_E\,K^{ED}\right](\Phi+\bar\Psi_0),$$

while the second order term gives the same expression evaluated in $\bar\Psi_0$ plus corrections of $O((\Phi^*)^3)$. Collecting all these expressions, the complete form of the nonlocal proper solution arises up to second order in the antifields

$$\begin{aligned} S_\Lambda(\Phi,\Phi^*) &= \tilde{\mathcal{S}}(\Phi,\bar\Psi_0) + \Phi^*_A(\varepsilon^2)^A_B R^B(\Phi+\bar\Psi_0) \\ &\quad + \frac{1}{2}\Phi^*_A\,(\varepsilon^2)^A_C\,\Phi^*_B\,(\varepsilon^2)^B_D\,\left[R^{DC} + R^D_F K^{FE} R^C_E\right](\Phi+\bar\Psi_0) + \ldots, \end{aligned} \tag{3.14}$$

with $R^A_B = \frac{\partial_l R^A(\Phi)}{\partial \Phi^B}$.

The lowest order antifield terms in (3.14) are precisely the (gauge-fixed) nonlocal action (2.9) and its nonlocal BRST transformations (2.13), as required. Properness of S_Λ comes now from the fact that the new interaction terms, of $O(\frac{1}{\Lambda^2})$, appearing in the (gauge-fixed) nonlocal action (2.9) can not alter the maximum rank of the hessian of the original gauge-fixed action (2.1). Continuing in this way, the quadratic piece in antifields provides the regulated form of the on-shell nilpotency structure functions, whose form could also have been inferred from computation (2.14) when dealing with an open algebra. In conclusion, up to this point the results of sect. 2 are completely reproduced in the FA framework. Further determination of the regulated form of the higher order structure coefficients would proceed in the same fashion, that is, by solving (3.8) at second and higher order in antifields and plugging the solution in (3.7).

In summary, expression (3.7) supplemented with (3.8) encodes in a compact way all the information about the structure of the nonlocally regulated BRST symmetry and naturally provides a nonlocally regularized proper solution to be used in algebraic perturbative computations. As in the original formulation, in the limit $\Lambda^2 \to \infty$, the nonlocally regulated gauge-fixed action $\tilde{\mathcal{S}}(\Phi, \bar{\Psi}_0)$ becomes the original one $\mathcal{S}(\Phi)$, while the antifield dependent part of (3.14) acquires the original form as well, as a consequence of the limits (2.10). In the unregulated limit thus (3.14) converges to the original proper solution (1.3) and the original local theory is recovered. This concludes the characterization of the regulated antibracket-antifield formalism at classical level.

3.2 Nonlocally regulated quantum master equation

From the interplay between the nonlocal regularization method and the antibracket-antifield formalism, a nonlocally regularized proper solution S_Λ of the classical master equation arises. In this sense, the classical BRST structure can be considered to be preserved at regularized level. Quantization of the theory, however, as indicated by the BRST Ward identity (1.6), requires in general the addition of extra counterterms or interactions M_p in order to preserve the quantum counterpart of the classical BRST structure, i.e., the substitution of the classical action S (1.3) by a suitable quantum action W (1.5). This fact was already stressed in the original references [1, 2, 3], although that approach seemed to suggest that only one-loop corrections M_1 would eventually suffice to acquire BRST invariance. Instead, the FA formalism indicates that generally two and higher order loop corrections should also be considered. The question arises then of how regularization of these new interaction terms, or conversely, of the new theory described by W, should proceed.

The way in which nonlocalization acts on the free part $F(\Phi)$ and on the complete interaction part $\mathcal{I}(\Phi, \Phi^*)$ of the original proper solution, defined as

$$\mathcal{I}(\Phi, \Phi^*) \equiv I(\Phi) + \Phi^*_A R^A(\Phi) + \frac{1}{2}\Phi^*_A \Phi^*_B R^{BA}(\Phi) + \ldots,$$

provides also a definite way to regularize interactions coming from counterterms M_p. Indeed, the process described in the previous section can be summarized by first constructing the auxiliary free and interaction parts

$$\tilde{F}(\Phi, \Psi) = F(\hat{\Phi}) - A(\Psi), \qquad \tilde{\mathcal{I}}(\Phi, \Phi^*, \Psi, \Psi^*) = \mathcal{I}(\Theta, \Theta^*),$$

with $\{\Theta, \Theta^*\}$ the linear combinations defined in (3.1), (3.2), and by putting afterwards the auxiliary fields Ψ on-shell and its antifields to zero

$$F_\Lambda(\Phi, \Phi^*) = \tilde{F}(\Phi, \bar{\Psi}_0), \qquad \mathcal{I}_\Lambda(\Phi, \Phi^*) = \tilde{\mathcal{I}}(\Phi + \bar{\Psi}_0, \Phi^* \varepsilon^2),$$

so that in the end $S_\Lambda = F_\Lambda + \mathcal{I}_\Lambda$.

The nonlocalization procedure for the quantum action W (1.5), expressed as

$$W = F + \mathcal{I} + \sum_{p=1}^{\infty} \hbar^p M_p \equiv F + \mathcal{Y}, \tag{3.15}$$

which defines the generalized "quantum" interaction $\mathcal{Y}$, should then proceed along the same lines, i.e., constructing first an auxiliary quantum action $\tilde{W}$

$$\begin{aligned} \tilde{W}(\Phi, \Phi^*, \Psi, \Psi^*) &= F(\hat{\Phi}) - A(\Psi) + \mathcal{Y}(\Theta, \Theta^*) \\ &= W(\Theta, \Theta^*) + \frac{1}{2}\Sigma^A \left[\frac{\mathcal{F}}{\varepsilon^2} + \frac{\mathcal{F}}{(1-\varepsilon^2)}\right]_{AB} \Sigma^B, \end{aligned} \tag{3.16}$$

and eliminating afterwards the auxiliary fields and antifields in the usual way, i.e.,

$$W_\Lambda(\Phi,\Phi^*) = \tilde{W}(\Phi,\Phi^*,\bar{\Psi}_q,\Psi^*=0), \tag{3.17}$$

where the quantum on-shell auxiliary fields $\bar{\Psi}_q$ are now the solutions of the equations of motion associated with the auxiliary quantum action $\tilde{W}$ (3.16)

$$\frac{\partial_r \tilde{W}(\Phi,\Phi^*;\Psi,\Psi^*=0)}{\partial \Psi^A} = 0 \quad \Rightarrow \quad \Psi^A = \left(\frac{\partial_r \mathcal{Y}}{\partial \Phi^B}(\Phi+\Psi,\Phi^*\varepsilon^2)\right)\mathcal{O}^{BA}. \tag{3.18}$$

The quantum on-shell auxiliary fields $\bar{\Psi}_q$ solving (3.18) are seen afterwards to contain as lowest order term in $\hbar$ the classical on-shell shadow fields $\bar{\Psi}$ verifying (3.8). This fact, together with the form (1.5) of W, indicates that W_Λ contains still the nonlocal proper solution S_Λ (3.7) as lowest order term in $\hbar$, while the higher order terms in $\hbar$ should be interpreted as the nonlocal form $M_{p,\Lambda}$ of the counterterms M_p, i.e.,

$$W_\Lambda = S_\Lambda + \sum_{n=1} \hbar^p M_{p,\Lambda}.$$

Therefore, the nonlocalization procedures for S and W coincide at tree (classical) level, as expected.

The nonlocally regulated quantum action W_Λ (3.17) obtained in this way is thus the object to be used in the path integral (1.4) in order to perform regularized perturbative computations. Defining afterwards Γ_Λ as the corresponding regulated effective action and following the same steps used to derive (1.6), which now seems to be meaningful, the regulated BRST Ward identity comes out

$$\frac{1}{2}(\Gamma_\Lambda,\Gamma_\Lambda) = -i\hbar(\mathcal{A}_\Lambda \cdot \Gamma_\Lambda), \tag{3.19}$$

where the obstruction $\mathcal{A}_\Lambda$ acquires now the form (1.7) with the substitution $W \to W_\Lambda$

$$\mathcal{A}_\Lambda = \left[\Delta W_\Lambda + \frac{i}{2\hbar}(W_\Lambda,W_\Lambda)\right]. \tag{3.20}$$

Let us now investigate more precisely the form (3.20) of the quantum master equation in this regularized framework. Consider first of all the action of the operator Δ on W_Λ

$$\Delta W_\Lambda = \frac{\partial_r\partial_l W_\Lambda}{\partial\Phi^B\partial\Phi^*_A}\delta^B_A.$$

Use of the analogs of relations (3.9) between derivatives of W_Λ and $\tilde{W}$,

$$\frac{\partial_r W_\Lambda}{\partial\Phi^A} = \left.\frac{\partial_r\tilde{W}}{\partial\Phi^A}\right|_q, \qquad \frac{\partial_l W_\Lambda}{\partial\Phi^*_A} = \left.\frac{\partial_l\tilde{W}}{\partial\Phi^*_A}\right|_q, \tag{3.21}$$

–where now the "q"-restriction means on the surface $\{\Psi=\bar{\Psi}_q(\Phi,\Phi^*),\Psi^*=0\}$– and of the explicit form (3.16) of $\tilde{W}$ in terms of the original W, which indicates that dependence of $\left.\frac{\partial_l\tilde{W}}{\partial\Phi^*_A}\right|_q$ on the fields Φ always appears through the combination $(\Phi+\bar{\Psi}_q)$, yields

$$\Delta W_\Lambda = \left[W^A_B(\Phi+\bar{\Psi}_q,\Phi^*\varepsilon^2)\,\frac{\partial_r(\Phi+\bar{\Psi}_q)^B}{\partial\Phi^C}\,(\varepsilon^2)^C_A\right],$$

with $W^A_B(\Phi,\Phi^*)$ given in terms of the original quantum action W by

$$W^A_B(\Phi,\Phi^*) = \frac{\partial_r\partial_l W}{\partial\Phi^B\partial\Phi^*_A}. \tag{3.22}$$

On the other hand, differentiation of the quantum equations of motion for Ψ (3.18)

$$\frac{\partial_l \bar{\Psi}^A_q}{\partial \Phi^B} = \frac{\partial_l (\Phi + \bar{\Psi}_q)^C}{\partial \Phi^B} \left(\frac{\partial_l \partial_r \mathcal{Y}}{\partial \Phi^C \partial \Phi^D} (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2) \right) \mathcal{O}^{DA},$$

allows to solve for the derivative of $\bar{\Psi}_q$ with respect Φ and write

$$\Delta W_\Lambda(\Phi, \Phi^*) = \left[W^A_B \, \mathcal{K}^{BC} (\mathcal{O}^{-1})_{CD} \, (\varepsilon^2)^D_A \right] (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2), \tag{3.23}$$

where the (inverse of the) operator $\mathcal{K}^{AB}$ is expressed as

$$(\mathcal{K}^{-1})_{AB}(\Phi, \Phi^*) = \left[(\mathcal{O}^{-1})_{AB} - \mathcal{Y}_{AB}(\Phi, \Phi^*) \right], \qquad \text{with} \qquad \mathcal{Y}_{AB} = \frac{\partial_l \partial_r \mathcal{Y}}{\partial \Phi^A \partial \Phi^B}.$$

The operator $(\mathcal{K}^{-1})_{AB}$ appears in this way as the natural quantum, antifield extension of the original operator $(K^{-1})_{AB}$ (2.15). Further rewriting of the operator $\mathcal{K}^{AC}(\mathcal{O}^{-1})_{CB}$ in (3.23) as

$$\mathcal{K}^{AC} (\mathcal{O}^{-1})_{CB} = \left[\mathcal{O}^{AC} (\mathcal{K}^{-1})_{CB} \right]^{-1} = \left(\delta^A_B - \mathcal{O}^{AC} \mathcal{Y}_{CB} \right)^{-1} \equiv (\delta_q)^A_B \tag{3.24}$$

finally yields

$$\Delta W_\Lambda(\Phi, \Phi^*) = \left[W^A_B \, (\delta_q)^B_C \, (\varepsilon^2)^C_A \right] (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2) \equiv \Omega(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2). \tag{3.25}$$

Comparing then with the original (formal) computation of ΔW

$$\Delta W = W^A_B \delta^B_A, \tag{3.26}$$

it is seen that nonlocal regularization acts by essentially distorting the identity δ^A_B in (3.26) to a regulated expression $(\delta_q)^A_C (\varepsilon^2)^C_B$ and by changing afterwards the arguments (Φ, Φ^*) of the resulting quantity $\Omega(\Phi, \Phi^*)$ to $(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2)$. In this sense, it proceeds in quite the same fashion as usual regularization methods (Fujikawa regularization, PV,...), i.e., by substituting in an appropriate way δ^A_B by some other suitable expression.

On the other hand, the second term in the regulated quantum master equation (3.20), namely (W_Λ, W_Λ), can be written, using relations (3.21), and in analogy with equation (3.10) for S_Λ, as

$$\frac{1}{2}(W_\Lambda, W_\Lambda) = \left. \frac{\partial_r \tilde{W}}{\partial \Phi^A} \frac{\partial_l \tilde{W}}{\partial \Phi^*_A} \right|_q = \left. \left(\frac{\partial_r \tilde{W}}{\partial \Phi^A} \frac{\partial_l \tilde{W}}{\partial \Phi^*_A} + \frac{\partial_r \tilde{W}}{\partial \Psi^A} \frac{\partial_l \tilde{W}}{\partial \Psi^*_A} \right) \right|_q = \frac{1}{2} \left. (\tilde{W}, \tilde{W}) \right|_q,$$

whereas use of the explicit form of $\tilde{W}$ (3.16) and the canonical character of the transformation (3.5) yields

$$\left. (\tilde{W}, \tilde{W}) \right|_q = (W, W)(\Theta, \Theta^*)|_q = (W, W)(\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2).$$

Taking thus into account the final form of ΔW_Λ (3.25) in terms of the quantity Ω and the above result, expression (3.20) adopts the form

$$\mathcal{A}_\Lambda = \left[\Omega + \frac{i}{2\hbar} (W, W) \right] (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2). \tag{3.27}$$

The regularized quantum master equation consists thus in a quantity, the one in square brackets, entirely computed in terms of the original theory, whereas all the dependence on the quantum on-shell auxiliary fields $\bar{\Psi}_q$ appears only in the argument. This fact simplifies considerably the study of this equation, so that in the end the knowledge of the precise form of these quantum on-shell shadow fields results to be unnecesary.

3.3 Anomalies

In this section, we will be mainly interested in the characterization of anomalies in this framework. The anomaly question in this context has already been considered in [18] for the specific examples of the chiral Schwinger model and QED. In what follows, this topic is adressed in a general way by investigating the form of the regularized quantum master equation (3.27).

In the standard, locally regularized approaches, genuine anomalies are interpreted as obstructions to the local solvability of the (complete) quantum master equation. Nonlocal regularization, however, due to its nonlocal character, must be supplemented with an interpretation of what is meant by "local solvability". A general criterion comes from the fact that nonlocality of the approach is only part of the regularization method, so that for a local, renormalizable theory, in the limit $\Lambda^2 \to \infty$, all the divergent and finite quantities arising upon computation of ΔW_Λ in (3.25) should become local. Therefore, acceptable solutions for W_Λ, or equivalently, for the counterterms $M_{p,\Lambda}$, in this framework must be entirely constructed, before taking the limit $\Lambda^2 \to \infty$, from the smearing operator ε^A_B (2.2) in such a way that any sort of nonlocality disappear in the unregularized limit. If such a choice does not exist, some of the original, local symmetries may become anomalous.

The regulated quantum master equation (3.27) can in fact be naturally decomposed into its divergent part and its finite part when $\Lambda^2 \to \infty$, the latter being indicated as $[\mathcal{A}_\Lambda]_0$. "Local" solvability of the divergent part, as already stressed in the original references [1, 2, 3], defines an interesting problem on its own, which is not going to be adressed in the present study. It is worth to note, however, that the validity of this regularization method, in view of the renormalization of the theory, heavily relies on the fulfillment of this condition.

Anomaly issues, instead, are encoded in the finite part, $[\mathcal{A}_\Lambda]_0$, of (3.27). More concretely, the expression of the anomaly predicted by the regularized BRST Ward identity (3.19) should be considered, as in other regularization approaches, the value of this finite part in the limit $\Lambda^2 \to \infty$

$$\mathcal{A} \equiv \lim_{\Lambda^2 \to \infty} \left[\Omega + \frac{i}{2\hbar}(W, W)\right]_0 (\Phi + \bar{\Psi}_q, \Phi^* \varepsilon^2),$$

which, after taking into account in the arguments the limits $\bar{\Psi}_q \to 0$, $\varepsilon^2 \to 1$ when $\Lambda^2 \to \infty$, becomes

$$\mathcal{A} = \left[(\Delta W)_R + \frac{i}{2\hbar}(W, W)\right] (\Phi, \Phi^*), \tag{3.28}$$

with the regularized value of ΔW defined as

$$(\Delta W)_R \equiv \lim_{\Lambda^2 \to \infty} [\Omega]_0, \tag{3.29}$$

and where, from now on, W will stand for the finite part of the actual quantum action (1.5), i.e., the one without the divergent parts of the counterterms needed for renormalization. In this way, as anticipated, due to the precise dependence of the regulated quantum master equation in the quantum on-shell shadow fields, computation of the anomaly can be performed without the precise knowledge of their form.

For practical perturbative calculations, it is convenient to analyze the $\hbar$ expansion of (3.28) in order to recognize the expressions of the p-loop obstructions appearing in the regularized BRST Ward identity (3.19). Consider then definition (3.25) for Ω. The operator $(\delta_q)^A_B$ (3.24) involved in its computation can first be written, using (3.15), as

$$(\delta_q)^A_B = \left[(\delta_\Lambda^{-1})^A_B - \sum_{p=1}^{\infty} \hbar^p (\mathcal{O} M_p)^A_B\right]^{-1},$$

where $(\delta_\Lambda)^A_B$ is defined by

$$(\delta_\Lambda)^A_B = \left(\delta^A_B - \mathcal{O}^{AC}\mathcal{I}_{CB}\right)^{-1} = \delta^A_B + \sum_{n=1}\left(\mathcal{O}^{AC}\mathcal{I}_{CB}\right)^n, \quad \text{with} \quad \mathcal{I}_{AB} = \frac{\partial_l\partial_r\mathcal{I}}{\partial\Phi^A\partial\Phi^B}, \tag{3.30}$$

and where $(\mathcal{O}M_p)^A_B$ stands for the shorthand notation

$$(\mathcal{O}M_p)^A_B = \mathcal{O}^{AC}(M_p)_{CB}, \quad \text{with} \quad (M_p)_{AB} = \frac{\partial_l\partial_r M_p}{\partial\Phi^A\partial\Phi^B}.$$

Further expansion in powers of $\hbar$ results finally in

$$(\delta_q)^A_B = (\delta_\Lambda)^A_B + \sum_{p=1}\hbar^p \sum_{\substack{0<p_1,\ldots,p_j\leq p\\ p_1+\ldots+p_j=p}} \left(\delta_\Lambda(\mathcal{O}M_{p_1})\delta_\Lambda\ldots\delta_\Lambda(\mathcal{O}M_{p_j})\delta_\Lambda\right)^A_{\ B}, \tag{3.31}$$

where the sum runs over all permutations of the indices $p_1,\ldots,p_j$ such that their sum is p, the corresponding power of $\hbar$.

Plugging now the $\hbar$ expansion of W^A_B (3.22), inferred from expansion (1.5) for W,

$$W^A_B = S^A_B + \sum_{p=1}^{\infty}\hbar^p(M_p)^A_B, \tag{3.32}$$

and expansion (3.31) in expression (3.25) for ΔW_Λ allows to recognize the different coefficients in the $\hbar$ expansion of Ω, $\Omega = \sum_0 \hbar^p\Omega_p$. The lowest order term, Ω_0, appears to be

$$\Omega_0 = \left[S^A_B(\delta_\Lambda)^B_C(\varepsilon^2)^C_A\right], \tag{3.33}$$

while for Ω_p, it is

$$\begin{aligned}\Omega_p &= \left[(M_p)^A_B(\delta_\Lambda)^B_C(\varepsilon^2)^C_A\right]\\ &+ \sum_{\substack{0<p_1,\ldots,p_j\leq p\\ p_1+\ldots+p_j=p}} \left[S^A_B\left(\delta_\Lambda(\mathcal{O}M_{p_1})\delta_\Lambda\ldots\delta_\Lambda(\mathcal{O}M_{p_j})\delta_\Lambda\right)^B_{\ C}(\varepsilon^2)^C_A\right]\\ &+ \sum_{\substack{0<p_1,\ldots,p_j< p\\ p_1+\ldots+p_j=p}} \left[(M_{p_1})^A_B\left(\delta_\Lambda(\mathcal{O}M_{p_2})\delta_\Lambda\ldots\delta_\Lambda(\mathcal{O}M_{p_j})\delta_\Lambda\right)^B_{\ C}(\varepsilon^2)^C_A\right].\end{aligned} \tag{3.34}$$

By finally dropping out the divergent part of these expressions as part of the renormalization procedure and taking the limit $\Lambda^2\to\infty$ in the remaining finite expressions, as indicated by (3.29), what should be taken as the regularized values of ΔS and ΔM_p, $(\Delta S)_R$ and $(\Delta M_p)_R$, is obtained

$$(\Delta S)_R \equiv \lim_{\Lambda^2\to\infty}[\Omega_0]_0, \qquad (\Delta M_p)_R \equiv \lim_{\Lambda^2\to\infty}[\Omega_p]_0, \quad p\geq 1. \tag{3.35}$$

These are precisely the values appearing in the regularized analogs of expressions (1.9) and (1.10), obtained from (3.28) upon expanding in $\hbar$

$$\mathcal{A}_1 = (\Delta S)_R + i(M_1, S), \tag{3.36}$$

$$\mathcal{A}_p = (\Delta M_{p-1})_R + \frac{i}{2}\sum_{q=1}^{p-1}(M_q, M_{p-q}) + i(M_p, S), \qquad p\geq 2, \tag{3.37}$$

which should be considered the form of the p-loop anomaly provided by the nonlocally regularized BRST Ward identity (3.19).

Some comments are finally in order about the above expressions. First of all, it is not difficult to see that $(\Delta S)_R$ and, as a consequence, the one-loop anomaly (3.36), satisfy the usual Wess-Zumino consistency condition [19]. Indeed, consider the quantity ΔS_Λ, which can be written in terms of Ω_0 (3.33) as

$$\Delta S_\Lambda(\Phi, \Phi^*) = \Omega_0(\Phi + \bar{\Psi}_0, \Phi^* \varepsilon^2),$$

with $\bar{\Psi}_0$ the classical on-shell shadow fields. The algebraic definitions of the antibracket (1.2) and of the operator Δ (1.8), together with the fact that S_Λ verifies the classical master equation (3.10), allow then to conclude the condition

$$(\Delta S_\Lambda, S_\Lambda)(\Phi, \Phi^*) = 0 \Rightarrow (\Omega_0, S)(\Phi + \bar{\Psi}_0, \Phi^* \varepsilon^2) = 0,$$

where in writing the right hand side, relations (3.9) and $\left.\frac{\partial_l \tilde{S}}{\partial \Psi^*_A}\right| = (\bar{\Psi}_0, S_\Lambda)$ have been used. This condition holds as well for the finite part of Ω_0, $[\Omega_0]_0$, so that in the limit $\Lambda^2 \to \infty$ the consistency condition arises

$$\lim_{\Lambda^2 \to \infty} ([\Omega_0]_0, S)(\Phi + \bar{\Psi}_0, \Phi^* \varepsilon^2) = ((\Delta S)_R, S)(\Phi, \Phi^*) = 0.$$

Nonlocal regularization is thus a consistent regularization procedure, in the sense that it yields one-loop anomalies which verify the Wess-Zumino consistency condition.

On the other hand, expression (3.37) for the higher order loop terms in the quantum master equation can naturally be interpreted, as already suggested in the introduction, as the one-loop corrections generated by the counterterms M_k, $k < p$, to the p-loop anomaly. The fact that $(\Delta M_p)_R$ (3.34) appears as a functional trace over a certain operator, one of the most typical characteristics of one-loop corrections, further supports this conclusion. The obtained results indicate then that the BRST Ward identity (1.6) and its regulated version (3.19) proposed in the FA framework are incomplete and that, at two and higher loops, fundamental pieces are missed in the resulting quantum master equation. This incompleteness, however, should only be considered as a deficiency in the derivation of the right hand side of the BRST Ward identity (1.6) and not as an inability of nonlocal regularization to deal with two and higher loop corrections. These conclusions are strongly supported by the example presented below, for which computation of the universal two-loop anomaly through the BRST variation of the nonlocally regulated effective action Γ leads to the right answer, while use of the two-loop quantum master equation reproduces only the contribution to the universal two-loop anomaly of a standard one-loop counterterm absorbing part of the one-loop anomaly.

4 An example: chiral W_3 Gravity

In this section, the use of the nonlocally regulated FA formalism is exemplified by calculating the one and two loop anomalies for chiral W_3 gravity in this framework. The forms obtained for these quantities are in complete agreement with previous results obtained in the literature [20, 21, 22, 23, 24], thereby showing the nonlocally regulated FA formalism as a suitable candidate for further study and characterization of the structure of higher loop BRST anomalies. We closely follow the conventions and notations of ref. [8], to which the reader is referred for the explicit construction of the proper solution of the master equation in the classical and gauge-fixed basis.

4.1 Nonlocalization of the proper solution

Chiral W_3 gravity [25] consists in a system of D scalar fields ϕ^i, $i = 1, \ldots, D$, coupled to gauge fields h and B through the spin-2 and spin-3 currents

$$T = \frac{1}{2}(\partial\phi^i)(\partial\phi^i), \qquad W = \frac{1}{3}d_{ijk}(\partial\phi^i)(\partial\phi^j)(\partial\phi^k), \tag{4.1}$$

where we are using the notations

$$\partial = \partial_+, \qquad \bar\partial = \partial_-, \qquad x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm x^0),$$

and where d_{ijk} is a constant, totally symmetric tensor satisfying the identity

$$d_{i(jk}d_{l)mi} = k\delta_{(jl}\delta_{k)m},$$

in terms of an arbitrary, but fixed parameter k. The classical action obtained in this way

$$S_0 = \int \mathrm{d}^2x \left[-\frac{1}{2}(\partial\phi^i)(\bar\partial\phi^i) + h\,T + B\,W\right],$$

with $\mathrm{d}^2x = \mathrm{d}x^0\mathrm{d}x^1 = \mathrm{d}x^+\mathrm{d}x^-$, is invariant under well-known spin-2 and spin-3 gauge symmetries having an open algebra.

In the gauge-fixed basis of fields and antifields, the proper solution of the classical master equation for this system, as constructed in [8], is given by[4]

$$\begin{aligned} S = \int \mathrm{d}^2x \quad & \Big\{\Big[-\frac{1}{2}(\partial\phi^i)(\bar\partial\phi^i) + b(\bar\partial c) + v(\bar\partial u)\Big] \\ & +\phi^*_i\left[c(\partial\phi^i) + ud_{ijk}(\partial\phi^j)(\partial\phi^k) - 2kb(\partial u)u(\partial\phi^i)\right] \\ & +b^*\left[-T + 2b(\partial c) + (\partial b)c + 3v(\partial u) + 2(\partial v)u\right] \\ & +v^*\left[-W + 2kTb(\partial u) + 2k\partial(Tbu) + 3v(\partial c) + (\partial v)c\right] \\ & +c^*\left[(\partial c)c + 2kT(\partial u)u\right] + u^*\left[2(\partial c)u - c(\partial u)\right]\Big\} \\ = \quad & \mathcal{S}(\Phi) + \Phi^*_A R^A(\Phi), \end{aligned} \tag{4.2}$$

where $\{c, u\}$ are the ghosts corresponding to spin-2 and spin-3 gauge symmetries; $\{b, v\}$, their associated antighosts; and $\{\phi^*_i, c^*, u^*, b^*, v^*\}$, the corresponding antifields. This action is obtained by first constructing the proper solution in the classical basis of fields and antifields $\{\phi^i, h, B, c, u; \phi^*_i, h^*, B^*, c^*, u^*\}$ and performing afterwards the canonical transformation to the gauge-fixed basis

$$\{h, h^*, B, B^*\} \to \{b = h^*, b^* = -h, v = B^*, v^* = -B\}.$$

It is worth to note that the resulting gauge-fixed action contains no interaction, i.e., $\mathcal{S}(\Phi) = F(\Phi)$ and $I(\Phi) = 0$, so that the requirement for applying nonlocal regularization to this model is satisfied. Interactions can be considered to be contained in the antifield dependent part, $\Phi^*_A R^A(\Phi)$, antifields acting then as a sort of coupling constants on which expansions can be performed.

The first step towards nonlocalization of the proper solution (4.2) is the identification of the kinetic operator $\mathcal{F}_{AB}$ in (2.1) for the propagating fields $\Phi^A = \{\phi^i; b, v; c, u\}$. In this basis, in which from now on all matrices are going to be expressed, its explicit expression reads

$$\mathcal{F}_{AB} = \begin{pmatrix} \partial\bar\partial\,\delta_{ij} & 0 & 0 \\ 0 & 0 & \mathbb{1}\,\bar\partial \\ 0 & \mathbb{1}\,\bar\partial & 0 \end{pmatrix},$$

[4]The free parameter α considered in [8] is taken here equal to 0 for simplicity.

where $\mathbb{1}$ stands for the identity in the spin 2 (spin 3) ghost sector. Introducing then an operator $(T^{-1})^{AB}$ of the form

$$(T^{-1})^{AB} = \begin{pmatrix} \delta^{ij} & 0 & 0 \\ 0 & 0 & \mathbb{1}\,\partial \\ 0 & \mathbb{1}\,\partial & 0 \end{pmatrix},$$

a suitable regulator, quadratic in space-time derivatives, arises

$$\mathcal{R}^A_B = (T^{-1})^{AC}\mathcal{F}_{CB} = \partial\bar{\partial}\,\delta^A_B, \tag{4.3}$$

with δ^A_B the identity in the complete space of fields. The corresponding smearing and shadow kinetic operator are afterwards constructed from (4.3) using the general expressions (2.2) and (2.3), resulting in

$$\varepsilon^A_B = \exp\left(\frac{\partial\bar{\partial}}{2\Lambda^2}\right)\delta^A_B \equiv \varepsilon\,\delta^A_B, \qquad \mathcal{O}^{-1}_{AB} = (\varepsilon^2-1)^{-1}\mathcal{F}_{AB}, \tag{4.4}$$

whereas $\mathcal{O}^{AB}$ takes the form

$$\mathcal{O}^{AB} = \begin{pmatrix} \mathcal{O} & 0 & 0 \\ 0 & 0 & \mathbb{1}\mathcal{O}\partial \\ 0 & \mathbb{1}\mathcal{O}\partial & 0 \end{pmatrix}, \qquad \text{with} \qquad \mathcal{O} \equiv \frac{(\varepsilon^2-1)}{\partial\bar{\partial}} = \int_0^1 \frac{\mathrm{d}t}{\Lambda^2}\exp\left(t\frac{\partial\bar{\partial}}{\Lambda^2}\right). \tag{4.5}$$

Nonlocalization of the proper solution (4.2) (or of a suitable quantum extension W of it, if counterterms are needed) would now proceed as described in section 2, that is, by introducing the shadow fields and antifields $\{\Psi^A, \Psi^*_A\}$, constructing from them and the above objects the auxiliary proper solution (3.4) (or its quantum extension (3.16)), and substituting the shadow fields by the solutions of their equations of motion, while putting their antifields to zero. In the present case, due to the absence of the classical interaction term $I(\Phi)$ and the highly nonlinear form of the BRST transformations $R^A(\Phi)$ in (4.2), it is clear that the classical shadow fields should be solved perturbatively in antifields. However, as analyzed in the previous section, computation of anomalies from the regularized form of the quantum master equation can be completely performed without this information, whereas the two-loop anomaly calculation, relying on diagrammatics, only needs, as indicated in section 2, the form of the auxiliary proper solution (3.4). For this reason, we skip this calculation and leave it as an exercise for the interested reader.

4.2 One-loop anomaly

Let us now calculate the one-loop anomaly according to the prescription (3.36) presented in the previous section. The main ingredients to compute $(\Delta S)_R$ are, as indicated by (3.35), (3.33), (3.30), the objects S^A_B, $\mathcal{O}^{AB}$, $\mathcal{I}_{AB}$ and $(\varepsilon^2)^A_B$.

The operators $(\varepsilon^2)^A_B$ and $\mathcal{O}^{AB}$ have been previously constructed and are given by (4.4) and (4.5), respectively. On the other hand, expressions for S^A_B, (3.32), (3.22), and $\mathcal{I}_{AB}$, (3.30), are constructed entirely in terms of the original proper solution S. In the present example, we have for S^A_B

$$S^A_B = \frac{\partial_r\partial_l S}{\partial\Phi^B\partial\Phi^*_A} =$$

$$\begin{pmatrix} c^i_j\partial & -2k(\partial u)u(\partial\phi^i) & 0 & (\partial\phi^i) & u^i \\ -(\partial\phi_j)\partial & -(c\partial)_2 & -2(u\partial)_{3/2} & (b\partial)_1 & 3(v\partial)_{1/3} \\ -u_j\partial & -2k[T(u\partial)_2+u(\partial T)] & -(c\partial)_3 & 3(v\partial)_{1/3} & 4k(bT\partial)_{1/2} \\ 2k(\partial u)u(\partial\phi_j)\partial & 0 & 0 & -(c\partial)_{-1} & -2kT(u\partial)_{-1} \\ 0 & 0 & 0 & -2(u\partial)_{-1/2} & -(c\partial)_{-2} \end{pmatrix}, \tag{4.6}$$

with T the spin-2 current in (4.1); c^i_j and u^i, the operators

$$
\begin{aligned}
c^i_j &= \left[c\delta^i_j - 2kb(\partial u)u\delta^i_j + 2ud^i_{\ jk}(\partial\phi^k)\right], \\
u^i &= d^i_{\ jk}(\partial\phi^j)(\partial\phi^k) - 2k\left[b(\partial u)(\partial\phi^i) + (b(\partial\phi^i)u\partial)_1\right],
\end{aligned}
$$

and where $(F(\Phi,\Phi^*)\,\partial)_n$ stands for the shorthand notation

$$
(F\,\partial)_n = F\,\partial + n(\partial F), \qquad (F\,\partial)^\dagger_n = -[F\,\partial + (1-n)(\partial F)] = -(F\,\partial)_{1-n}. \tag{4.7}
$$

In much the same way, the operator $\mathcal{I}_{AB}$ reads in this case

$$
\mathcal{I}_{AB} = \frac{\partial_l\partial_r}{\partial\Phi^A\partial\Phi^B}\left[\Phi^*_C R^C(\Phi)\right] =
$$

$$
\begin{pmatrix}
\partial\, h^*_{ij}\,\partial & (g^*_i\partial)_1 & 0 & -(\phi^*_i\partial)_1 & (q^*_i\partial)_1 \\
g^*_j\,\partial & 0 & 0 & (b^*\partial)_{-1} & r^* \\
0 & 0 & 0 & 2(v^*\partial)_{-1/2} & (b^*\partial)_{-2} \\
-\phi^*_j\,\partial & (b^*\partial)_2 & 2(v^*\partial)_{3/2} & 2(c^*\partial)_{1/2} & -3(u^*\partial)_{2/3} \\
q^*_j\,\partial & -(r^*)^\dagger & (b^*\partial)_3 & -3(u^*\partial)_{1/3} & 2(p^*\partial)_{1/2}
\end{pmatrix}, \tag{4.8}
$$

where the linear quantities in the antifields h^*_{ij}, b^*_i, q^*_i, r^* and p^* are given by

$$
\begin{aligned}
h^*_{ij} &= \left\{\delta_{ij}\left[b^* + 2kb(u(\partial v^*) - v^*(\partial u)) + 2kc^*u(\partial u)\right] - 2d_{ij}^{\ \ k}\phi^*_k u + 2v^* d_{ijk}(\partial\phi^k)\right\}, \\
g^*_i &= 2k\left[\phi^*_i(\partial u)u + (v^*(\partial u) - u(\partial v^*))(\partial\phi_i)\right], \\
q^*_i &= \left\{-2\phi^*_j d^j_{\ ik}(\partial\phi^k) + 2k\left[(\partial(v^*b(\partial\phi_i) + u\phi^*_i b)) + b(\partial\phi_j)(v^*\partial)_1 + \phi^*_i b(u\partial)_1\right]\right\}, \\
r^* &= 2k\left[T(v^*\partial)_{-1} - \phi^*_i(\partial\phi^i)(u\partial)_{-1}\right], \\
p^* &= 2k\left[Tc^* - \phi^*_i(\partial\phi^i)b\right].
\end{aligned}
$$

Linearity of the proper solution (4.2) in antifields leads thus to operators S^A_B (4.6) and $\mathcal{I}_{AB}$ (4.8) independent of and linear in the antifields, respectively, so that expression (3.33) for Ω_0 becomes an usual antifield expansion, after plugging in it expansion (3.30) for $(\delta_\Lambda)^A_B$ adapted to this case. Dimensional analysis in terms of the combination $d-j$ –"engineering" dimension minus spin– appears then to be very useful to figure out the relevant terms in this antifield expansion for Ω_0. Indeed, the value of the $d-j$ combination for the relevant quantities involved in its computation

$$
(d-j)[\Phi^A,\,\partial] = 0, \qquad (d-j)[\Phi^*_A,\,\Lambda^2] = 2, \tag{4.9}
$$

and the fact that $(\Delta S)_R \sim \Omega_0$ is the integral of a quantity of dimension $d=2$ and spin $j=0$, or $d-j=2$, constraint the possible terms arising in its calculation to be necessarily of the form

$$
\Lambda^{-2n}(\Phi^*)^{2(n+1)}F_n(\Phi;\partial), \qquad n = -1, 0, 1, \ldots,
$$

so that only the terms $n=-1,0$ –the divergent, antifield independent piece and the finite term, linear in antifields– are really relevant. Collecting then all these facts together, it is concluded that the relevant terms in expansion (3.33), (3.30) for Ω_0 are

$$
\Omega_0 = \left[\varepsilon^2\, S^A_A\right] + \left[\varepsilon^2\, S^A_B\,\mathcal{O}^{BC}\,\mathcal{I}_{CA}\right] + O\left(\frac{(\Phi^*)^2}{\Lambda^2}\right). \tag{4.10}
$$

The antifield independent term in expansion (4.10), encoding potential divergencies, is just the functional trace in the continuous indices of S^A_A weighted with the "damping" operator ε^2, resulting in

$$\left[\varepsilon^2\, S^A_A\right] = \mathrm{Tr}\left[\varepsilon^2\left(c^i_i\partial - (c\partial)_2 - (c\partial)_3 - (c\partial)_{-1} - (c\partial)_{-2}\right)\right] = \mathrm{Tr}\left[\varepsilon^2\left(c^i_i\partial - 4(c\partial)_{1/2}\right)\right] = 0,$$

since $\mathcal{T}(F,n) = \mathrm{Tr}\left[\varepsilon^2(F\partial)_n\right] = 0$, as stated by computation (A.1) in the appendix. The vanishing of this term indicates that, upon expanding in Λ^2, finite contributions to $[\Omega_0]_0$ will only come from the second term in (4.10).

The next step is then the determination of the diagonal elements of the matrix $S^A_B\mathcal{O}^{BC}\mathcal{I}_{CD}$. A straightforward calculation yields

$$\mathrm{diag}(S^A_B\mathcal{O}^{BC}\tilde{I}_{CD}) = (A^i_j,\, A^b_b\,,\, A^v_v\,,\, A^c_c\,,\, A^u_u),$$

with the above operators given by

$$\begin{aligned}
A^i_j &= c^{ik}\,\partial\,\mathcal{O}\,\partial\,h^*_{kj}\,\partial + 2k(\partial u)u(\partial\phi^i)\,\mathcal{O}\,\partial\,\phi^*_j\,\partial + (\partial\phi^i)\,\mathcal{O}\,\partial\,g^*_j\,\partial, && (4.11)\\
A^b_b &= -(c\partial)_2\,\mathcal{O}\,\partial\,(b^*\partial)_2 - (\partial\phi^i)\,\partial\,\mathcal{O}\,\partial\,g^*_i\,\partial + 2(u\partial)_{3/2}\,\mathcal{O}\,\partial\,(r^*)^\dagger, && (4.12)\\
A^v_v &= -(c\partial)_3\,\mathcal{O}\,\partial\,(b^*\partial)_3 - 4k\left[T(u\partial)_2 + u(\partial T)\right]\,\mathcal{O}\,\partial\,(v^*\partial)_{3/2}, && (4.13)\\
A^c_c &= -(c\partial)_{-1}\,\mathcal{O}\,\partial\,(b^*\partial)_{-1} - 2k(\partial u)u(\partial\phi^i)\,\partial\,\mathcal{O}\,(\phi^*_i\partial)_1 - 4k\,T(u\partial)_{-1}\,\mathcal{O}\,\partial\,(v^*\partial)_{-1/2}, && (4.14)\\
A^u_u &= -(c\partial)_{-2}\,\mathcal{O}\,\partial\,(b^*\partial)_{-2} - 2(u\partial)_{-1/2}\,\mathcal{O}\,\partial\,r^*. && (4.15)
\end{aligned}$$

The expression of $(\Delta S)_R$ will then be, according to (3.35)

$$(\Delta S)_R = \lim_{\Lambda^2\to\infty}[\Omega_0]_0 = \lim_{\Lambda^2\to\infty}\mathrm{Tr}\left[\varepsilon^2\left(A^i_i + A^b_b + A^v_v + A^c_c + A^u_u\right)\right]. \qquad (4.16)$$

At this point, it is interesting to note that all the above traces appear, or can be written, as particular cases of the general expression

$$\mathcal{T}(F,G;n,m) = \lim_{\Lambda^2\to\infty}\mathrm{Tr}\left[\varepsilon^2\,F\,\partial^n\,\mathcal{O}\,\partial\,G\,\partial^m\right], \qquad (4.17)$$

whose explicit computation, performed in the appendix, yields

$$\mathcal{T}(F,G;n,m) = \frac{-i}{2\pi}\left[\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+m+1-k}\left(1-\frac{1}{2^{n+m+1-k}}\right)\right]\int \mathrm{d}^2x\,F\,\partial^{n+m+1}\,G. \qquad (4.18)$$

This result can now be extensively used to calculate the functional trace (4.16). For example, the first term in A^i_j (4.11), corresponding to the matter fields, reads in the above notation $\mathcal{T}(c^{ij}, h^*_{ji}; 1, 1)$, giving in this way the well-known contribution

$$(\Delta S)^{(i)}_R = \frac{i}{24\pi}\int \mathrm{d}^2x\,c^{ij}\,\partial^3\,h^*_{ij}, \qquad (4.19)$$

which besides the usual W_2 one-loop anomaly $\frac{iD}{24\pi}\int \mathrm{d}^2x\,c\,\partial^3\,b^*$ contains some new terms in the rest of antifields. The remaining contributions which contain explicitly the fields c and b^*, coming from the ghosts field entries (4.12)–(4.15), share the generic form

$$\lim_{\Lambda^2\to\infty}\mathrm{Tr}\left[\varepsilon^2\,(c\partial)_j\,\mathcal{O}\,\partial\,(b^*\partial)_j\right] = j^2\mathcal{T}((\partial c),(\partial b^*);0,0) + \mathcal{T}(c,b^*;1,1)$$

$$+j\left[\mathcal{T}((\partial c),b^*;0,1) + \mathcal{T}(c,(\partial b^*);1,0)\right] = \frac{i}{24\pi}(6j^2-6j+1)\int \mathrm{d}^2x\,c\,\partial^3\,b^*,$$

where j is the spin of the field associated with the entry, i.e., $j = (2, 3, -1, -2)$. All together, they add up to the well-known result

$$(\Delta S)_R^{(ii)} = \frac{-100\, i}{24\pi} \int \mathrm{d}^2x\, c\, \partial^3\, b^*. \tag{4.20}$$

The remaining contributions contain either v^* or ϕ_i^*. Proceeding in much the same way by using the general result (4.18), the v^* terms are computed to be

$$(\Delta S)_R^{(iii)} + (\Delta S)_R^{(iv)} = \frac{ik}{2\pi} \int \mathrm{d}^2x\, (v^*(\partial u) - u(\partial v^*))(\partial\phi^i)(\partial^3\phi^i)$$
$$+\frac{ik}{6\pi} \int \mathrm{d}^2x\, T\left[5(\partial^3 u)v^* - 12(\partial^2 u)(\partial v^*) + 12(\partial u)(\partial^2 v^*) - 5u(\partial^3 v^*)\right] \quad , \tag{4.21}$$

while the ϕ_i^* contribution results in

$$(\Delta S)_R^{(v)} = \frac{-ik}{6\pi} \int \mathrm{d}^2x\, \phi_i^* \left[6\partial\left(u(\partial u)(\partial^2\phi^i)\right) + 9(\partial^2 u)(\partial u)(\partial\phi^i) + 8u(\partial^3 u)(\partial\phi^i)\right]. \tag{4.22}$$

In summary, expressions (4.19), (4.20), (4.21) and (4.22) together constitute the form of $(\Delta S)_R$, or of the complete consistent one-loop anomaly if no counterterm M_1 is considered, provided by the nonlocally regularized FA formalism. Its form is in complete agreement with the PV calculation performed in [8], for the case $\alpha = 0$, and with earlier results in the literature [21, 22, 23, 24], thus indicating that expression (3.36) correctly reproduces one-loop anomalies.

4.3 Counterterm corrections to the one and two-loop anomalies

Contributions $(\Delta S)_R^{(i)}$ (4.19) and $(\Delta S)_R^{(ii)}$ (4.20) to the complete expression of $(\Delta S)_R$ are seen to contain a universal (gravitational) one-loop anomaly, depending only on the spin 2 ghost c and antifield b^* (or gauge field h when working in the classical basis),

$$(\Delta S)_{R,\mathrm{univ}} = \frac{i(D-100)}{24\pi} \int \mathrm{d}^2x\, c\, \partial^3\, b^*, \tag{4.23}$$

a mixed spin 2-spin 3 anomaly[5]

$$(\Delta S)_{R,\mathrm{mix}} = \frac{ikD}{12\pi} \int \mathrm{d}^2x\, \left\{c\, \partial^3\, \left[b(u(\partial v^*) - v^*(\partial u)) + c^* u(\partial u)\right] - \left[b(\partial u)u\right] \partial^3\, b^*\right\} \tag{4.24}$$

plus some extra terms containing either matter fields ϕ^i and/or antifields ϕ_i^*, which add to contributions (4.21), (4.22) in order to define the complete matter dependent one-loop anomaly.

The appearence of the mixed spin 2-spin 3 anomaly (4.24) can be traced back [25] to the non-primary character of the total spin 3 current $W + W_{\mathrm{gh}}$ –appearing in (4.2) as the v^* coefficient– with respect the total energy momentum tensor $T + T_{\mathrm{gh}}$ –the b^* coefficient in (4.2). Such part of the anomaly, however, can be absorbed by modifying in a suitable way the definition of the total spin 3 current and of the BRST transformation for the ghost c [23, 24], which in the present notation amounts to add the finite one-loop counterterm

$$M_1 = \beta \int \mathrm{d}^2x\, \left\{v^*\left[2u(\partial^3 b) + 9(\partial u)(\partial^2 b) + 15(\partial^2 u)(\partial b) + 10(\partial^3 u)b\right]\right.$$
$$\left. +c^*\left[2u(\partial^3 u) - 3(\partial u)(\partial^2 u)\right]\right\}, \tag{4.25}$$

[5]It should be noted that the k^2–proportional terms present in $(\Delta S)_R^{(i)}$ (4.19) amount to a total derivative. Therefore they are dropped out from (4.24).

with $\beta = \frac{-kD}{192\pi}$, and define the quantum action (1.5) as $W = S + \hbar M_1$.

The addition of such counterterm results in the following modifications with respect the original anomalies given by S (4.2). On the one hand, the expression of the complete one-loop anomaly $\mathcal{A}_1$ changes according to (3.36). The net effect, apart from the absence of the mixed spin 2-spin 3 anomaly (4.24), is a modification of the contribution $(\Delta S)_R^{(iv)}$ in (4.21) to

$$\begin{aligned}(\widehat{\Delta S})_R^{(iv)} = \frac{ik}{6\pi}\int \mathrm{d}^2x\, T \ \Bigg[&\left(5+\frac{D}{16}\right)(\partial^3 u)v^* - \left(12+\frac{3D}{32}\right)(\partial^2 u)(\partial v^*) \\ &+ \left(12+\frac{3D}{32}\right)(\partial u)(\partial^2 v^*) - \left(5+\frac{D}{16}\right)u(\partial^3 v^*)\Bigg],\end{aligned}$$

and of the term $(\Delta S)_R^{(v)}$ (4.22) to

$$\begin{aligned}(\widehat{\Delta S})_R^{(v)} = \frac{-ik}{6\pi}\int \mathrm{d}^2x\, \phi_i^* \ \Bigg[&6\partial\left(u(\partial u)(\partial^2\phi^i)\right) + \left(9+\frac{3D}{32}\right)(\partial^2 u)(\partial u)(\partial \phi^i) \\ &+ \left(8+\frac{2D}{32}\right)u(\partial^3 u)(\partial\phi^i)\Bigg].\end{aligned}$$

In this way, the new one-loop anomaly $\mathcal{A}_1$ consists now in the universal term $(\Delta S)_{R,\mathrm{univ}}$ (4.23) plus some matter dependent contributions.

Radiative corrections induced by the one-loop counterterm M_1 (4.25), on the other hand, modify as well the two-loop anomaly, whose explicit expression is going to be calculated in the next section. In the nonlocally regularized FA formalism this modification is conjectured to be described by the two-loop quantum master equation

$$\mathcal{A}_2 = (\Delta M_1)_R + \frac{i}{2}(M_1, M_1) + i(M_2, S), \tag{4.26}$$

with $(\Delta M_1)_R$ given by (3.35) as $(\Delta M_1)_R \equiv \lim_{\Lambda^2\to\infty}[\Omega_1]_0$, and where the explicit expression of Ω_1 is read off from (3.34) to be

$$\Omega_1 = \left[(M_1)^A_B(\delta_\Lambda)^B_C(\varepsilon^2)^C_A\right] + \left[S^A_B\left(\delta_\Lambda(\mathcal{O}M_1)\delta_\Lambda\right)^B_C(\varepsilon^2)^C_A\right]. \tag{4.27}$$

The forthcoming calculation will serve as an explicit verification of this conjecture at two-loop order.

In the present case, the term (M_1, M_1) in (4.26) evidently vanishes. Without adding further two-loop counterterms M_2, the two-loop anomaly shift is thus completely contained in $\Omega_1 \sim (\Delta M_1)_R$, for whose explicit computation the quantities $(M_1)^A_B$ and $(M_1)_{AB}$, together with S^A_B, $\mathcal{O}^{AB}$, $\mathcal{I}_{AB}$ and $(\varepsilon^2)^A_B$ previously obtained, are needed. M_1 (4.25) being linear in antifields, it is then clear that the non-vanishing entries of $(M_1)^A_B$, $\{(M_1)^{v^*}_b, (M_1)^{v^*}_u, (M_1)^{c^*}_u\}$, and $(M_1)_{AB}$, $\{(M_1)_{ub}, (M_1)_{bu}, (M_1)_{uu}\}$, of which only the explicit expressions of $\{(M_1)^{v^*}_b, (M_1)^{c^*}_u, (M_1)_{ub}\}$ turn out to be necessary in the end for the present calculation

$$\begin{aligned}(M_1)^{v^*}_b &= \beta\left[2u\partial^3 + 9(\partial u)\partial^2 + 15(\partial^2 u)\partial + 10(\partial^3 u)\right] \equiv \beta\mathcal{L}(u), \\ (M_1)^{c^*}_u &= \beta\left[2u\partial^3 - 3(\partial u)\partial^2 + 3(\partial^2 u)\partial - 2(\partial^3 u)\right] \equiv -\beta\mathcal{L}^\dagger(u), \\ (M_1)_{ub} &= -\beta\mathcal{L}(v^*), \qquad (M_1)_{bu} = -(M_1)^\dagger_{ub} = \beta\mathcal{L}^\dagger(v^*),\end{aligned}$$

are antifield independent and linear in antifields, respectively. Further use of dimensional analysis in terms of the combination $d - j$ (4.9) and the fact that the integrand of $(\Delta M_1)_R$ should also

have dimension $d = 2$ and spin $j = 0$, or $d - j = 2$, together with the antifield expansions of the above objects, singles out the relevant terms in (4.27) to be

$$\Omega_1 = \left[\varepsilon^2 (M_1)^A_A\right] + \left[\varepsilon^2 (M_1)^A_B \mathcal{O}^{BC} \mathcal{I}_{CA}\right] + \left[\varepsilon^2 S^A_B \mathcal{O}^{BC} (M_1)_{CA}\right] + O\left(\frac{(\Phi^*)^2}{\Lambda^2}\right). \tag{4.28}$$

The antifield independent term in (4.28), describing potential divergencies, vanishes due to the absence of diagonal terms in $(M_1)^A_B$. Once again no divergencies appear and finite contributions to $[\Omega_1]_0$ are seen to come only from the remaining pieces in (4.28). The diagonal elements of the matrices involved in the computation of these terms

$$\begin{aligned}
\mathrm{diag}\left((M_1)^A_B \mathcal{O}^{BC} \mathcal{I}_{CD}\right) &= (0,\,0,\,B^v_v,\,B^c_c,\,0),\\
\mathrm{diag}\left(S^A_B \mathcal{O}^{BC} (M_1)_{CD}\right) &= (0,\,B^b_b,\,0,\,0,\,B^u_u),
\end{aligned}$$

results then to be

$$\begin{aligned}
B^b_b &= 2\beta (u\partial)_{3/2}\, \mathcal{O}\, \partial\, \mathcal{L}(v^*), &(4.29)\\
B^v_v &= 2\beta \mathcal{L}(u)\, \mathcal{O}\, \partial\, (v^*\partial)_{3/2}, &(4.30)\\
B^c_c &= -2\beta \mathcal{L}^\dagger(u)\, \mathcal{O}\, \partial\, (v^*\partial)_{-1/2}, &(4.31)\\
B^u_u &= -2\beta (u\partial)_{-1/2}\, \mathcal{O}\, \partial\, \mathcal{L}^\dagger(v^*), &(4.32)
\end{aligned}$$

yielding the following form of $(\Delta M_1)_R$

$$(\Delta M_1)_R = \lim_{\Lambda^2\to\infty} \mathrm{Tr}\left[\varepsilon^2 \left(B^b_b + B^v_v + B^c_c + B^u_u\right)\right].$$

Using then cyclicity of the trace, property (4.7) and the symmetries of the above expression under the interchange $u \leftrightarrow v^*$, the above expression can be written in the more convenient form

$$(\Delta M_1)_R = \lim_{\Lambda^2\to\infty} \left\{-4\beta\, \mathrm{Tr}\left[\varepsilon^2\, \mathcal{L}^\dagger(u)\, \mathcal{O}\, \partial\, (v^*\partial)_{-1/2}\right] + 4\beta\, \mathrm{Tr}\left[\varepsilon^2\, \mathcal{L}(u)\, \mathcal{O}\, \partial\, (v^*\partial)_{3/2}\right]\right\},$$

in which the first term groups together the contributions of the spin 2 ghost sector, i.e., the ones coming from the B^b_b (4.29) and B^c_c (4.31) entries, while the second term stands for the contribution of the spin 3 ghost sector, produced by the B^v_v (4.30) and B^u_u (4.32) entries.

Once again, the above functional traces can be computed by means of the general expression (4.18). In this way, the spin 2 and 3 ghost sector contributions are seen to be, respectively,

$$\mathcal{A}^{(2)}_2 = \frac{i\beta}{5\pi}\frac{199}{8} \int \mathrm{d}^2x\, u\, \partial^5 v^*, \qquad \mathcal{A}^{(3)}_2 = \frac{i\beta}{5\pi}\frac{149}{8} \int \mathrm{d}^2x\, u\, \partial^5 v^*,$$

which add up, after substituting the actual value of β, to the one-loop correction $\mathcal{A}_2$ (4.26) produced by the counterterm M_1 (4.25) to the two-loop anomaly

$$\mathcal{A}_2 = \frac{-i87Dk}{1920\pi^2} \int \mathrm{d}^2x\, u\, \partial^5 v^*, \tag{4.33}$$

in complete agreement, apart from numerical factors due to the difference in the used conventions, with previous results [24]. This computation constitutes thus a direct verification for $p = 2$ of the conjecture stated along the paper, namely, of the interpretation of expression (3.37) for the higher order terms of the quantum master equation as one-loop corrections to the p-loop anomaly generated by the counterterms M_k, $k < p$ and, consequently, of the incompleteness of the regulated quantum master equation (3.27) itself.

4.4 Universal two-loop Anomaly

In this section, in order to finally show that incompleteness of the quantum master equation comes from a naive derivation of the FA BRST Ward identity (1.6), or of its regulated version (3.19), and not as a drawback of the nonlocal regularized FA formalism by itself, the computation of the universal W_3 two-loop anomaly is performed by using the "left hand side" of the BRST Ward identity (3.19) that is, by first calculating the relevant part of the two-loop effective action in the nonlocally regulated FA framework and afterwards performing its BRST variation.

The relevant two-loop 1PI diagram, involving only matter fields ϕ^i in its internal lines and the antifields v^* ($v^* = -B$, when in the classical basis) as external sources (Fig. 2)

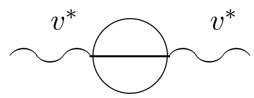

Fig. 2. Unregularized two-loop matter diagram.

comes from the contraction of the vertex $-\frac{1}{3}v^* d_{ijk}(\partial\phi^i)(\partial\phi^j)(\partial\phi^k)$ of the gauge fixed proper solution (4.2) with itself. Its unregularized contribution is read off from the Feynman rules obtained from (4.2) to be

$$\Gamma(v^*, v^*) = -\frac{i}{3}\hbar^2 d^2 \int \mathrm{d}^2x\, \mathrm{d}^2y\; v^*(x) \left[\partial \left(\frac{i}{\partial\bar{\partial}}\right) \partial\, \delta^2(x-y)\right]^3 v^*(y), \tag{4.34}$$

with $d^2 \equiv d_{ijk}d^{ijk}$.

As previously noticed in the original references [2, 3], when performing diagramatic calculations, the nonlocally regulated theory is effectively realized by using the auxiliary action (3.4) and eliminating by hand closed loops formed solely with barred lines. For the present calculation, the relevant terms in the auxiliary proper solution associated with (4.2), giving the modified propagators and the "regularized" version of the considered vertex, are

$$\begin{aligned}\tilde{S} \;\; &= \;\; \int \mathrm{d}^2x \left[\frac{1}{2}\phi^i \left(\frac{\partial\bar{\partial}}{\varepsilon^2}\right)\phi^i + \frac{1}{2}\psi^i \left(\frac{\partial\bar{\partial}}{1-\varepsilon^2}\right)\psi^i \right. \\ &\left. \qquad -\frac{1}{3}(\varepsilon^2 v^*) d_{ijk}(\partial(\phi^i+\psi^i))(\partial(\phi^j+\psi^j))(\partial(\phi^k+\psi^k)) + \ldots \right],\end{aligned}$$

where ψ^i stand for the shadow fields corresponding to the original matter fields ϕ^i. The new Feynman rules coming from this action lead then to the following $2^3 = 8$ diagrams with only internal matter lines and two external v^* lines (Fig. 3)

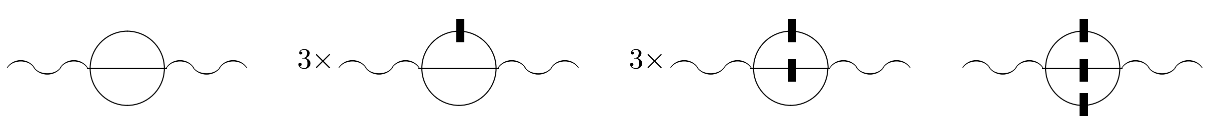

Fig. 3. Two-loop matter diagrams in the nonlocally regulated theory.

where, in the second and third diagrams "3×" stands for the three different diagrams obtained by combining the corresponding unbarred and barred propagators.

From them, and upon elimination of the last contribution formed solely with barred lines, the regularized expression of $\Gamma(v^*, v^*)$ (4.34) arises

$$\Gamma_\Lambda(v^*, v^*) = -\frac{i}{3}\hbar^2 d^2 \int \mathrm{d}^2x\, \mathrm{d}^2y\, \left(\varepsilon^2 v^*(x)\right) K(x, y; \Lambda^2) \left(\varepsilon^2 v^*(y)\right), \tag{4.35}$$

where the regularized expression of the kernel $K(x, y; \Lambda^2)$ in (4.35) reads, using the more convenient momentum representation

$$K(x, y; \Lambda^2) = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\, e^{ip\cdot(x-y)} \Bigg\{ (-i)^3 \int_{R_1^3} \frac{\mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3}{\Lambda^6} \int \frac{\mathrm{d}^2k_1}{(2\pi)^2} \int \frac{\mathrm{d}^2k_2}{(2\pi)^2}$$
$$(ik_1)^2\, (ik_2)^2\, (i(p-k_1-k_2))^2 \exp\left[-t_1 \frac{k_1\bar{k}_1}{\Lambda^2} - t_2 \frac{k_2\bar{k}_2}{\Lambda^2} - t_3 \frac{(p-k_1-k_2)(\bar{p}-\bar{k}_1-\bar{k}_2)}{\Lambda^2} \right] \Bigg\},$$

with R_1^3 denoting the standard 3-parameter region of integration at two-loops, namely, the first quadrant of R^3 minus the cube $[0, 1]^3$ [3].

The term in the exponential can now be "diagonalized", and integrations over momenta decoupled, by performing the standard change of variables

$$k_1^\mu \rightarrow \left[\frac{\Lambda}{(t_1+t_3)^{1/2}}\, k_1^\mu - \frac{\Lambda\, t_3}{(t_1+t_3)^{1/2}\, t_{123}^{1/2}}\, k_2^\mu + \frac{t_2 t_3}{t_{123}}\, p^\mu \right],$$
$$k_2^\mu \rightarrow \left[\frac{\Lambda (t_1+t_3)^{1/2}}{t_{123}^{1/2}}\, k_2^\mu + \frac{t_3 t_1}{t_{123}}\, p^\mu \right],$$

with the combination t_{123} defined as $t_{123} = (t_1t_2 + t_2t_3 + t_3t_1)$. Afterwards, in the polynomial in k_1, k_2 arising upon this change from the factor $\left[(ik_1)^2(ik_2)^2(i(p-k_1-k_2))^2\right]$ in the integrand, only the k-independent term survives, as deduced from integrals (A.2) in the appendix. Performing then the remaining gaussian integrals in k_1, k_2, the following result arises

$$K(x, y; \Lambda^2) = (-i)^3 \left(\frac{-i}{2\pi}\right)^2 \int \frac{\mathrm{d}^2p}{(2\pi)^2} \frac{(ip)^6}{\Lambda^2}\, e^{ip\cdot(x-y)} \mathcal{K}\left(\frac{p\bar{p}}{\Lambda^2}\right), \tag{4.36}$$

with the quantity $\mathcal{K}\left(\frac{p\bar{p}}{\Lambda^2}\right)$ given by

$$\mathcal{K}\left(\frac{p\bar{p}}{\Lambda^2}\right) = \int_{R_1^3} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3\, \frac{(t_1t_2t_3)^4}{t_{123}^7} \exp\left[-\frac{t_1t_2t_3}{t_{123}} \frac{p\bar{p}}{\Lambda^2} \right]. \tag{4.37}$$

This type of integrals are most easily performed by using the general change of variables in parameter space depicted in [3]. In the present case, that change amounts to $(t_1, t_2, t_3) \rightarrow (x, y, t)$ with x, y and t expressed as

$$t = t_1 + t_2 + t_3, \qquad x = \frac{t_2 + t_3}{t}, \qquad y = \frac{t_3}{t}.$$

In terms of these variables, the integral (4.37) becomes

$$\mathcal{K}\left(\frac{p\bar{p}}{\Lambda^2}\right) = 6 \int_0^{1/3} \mathrm{d}y \int_{2y}^{(1+y)/2} \mathrm{d}x \frac{\left[(1-x)(x-y)y\right]^4}{\left[x(1-x) + y(x-y)\right]^7}$$
$$\times \int_{1/(x-y)}^{\infty} \mathrm{d}t\, \exp\left[-t \left(\frac{(1-x)(x-y)y}{x(1-x) + y(x-y)} \frac{p\bar{p}}{\Lambda^2} \right) \right],$$

so that the integral over t is easily performed, producing an incomplete gamma function, defined as

$$\Gamma(n,z)=\int_z^\infty \mathrm{d}t\, t^{n-1}\, e^{-t},$$

and yielding

$$\begin{aligned}\mathcal{K}\left(\frac{p\bar{p}}{\Lambda^2}\right) &= 6\int_0^{1/3}\mathrm{d}y\int_{2y}^{(1+y)/2}\mathrm{d}x\,\frac{[(1-x)(x-y)y]^3}{[x(1-x)+y(x-y)]^6}\,\frac{\Lambda^2}{p\bar{p}}\\ &\quad\times\Gamma\left(1,\frac{y(1-x)}{x(1-x)+y(x-y)}\,\frac{p\bar{p}}{\Lambda^2}\right). \end{aligned}\tag{4.38}$$

Upon substitution now of (4.38) in the expression of the regularized kernel (4.36), the respective Λ^2 and Λ^{-2} factors cancel out. The resulting expression is then finite, so that the limit $\Lambda^2\to\infty$, which gives $\Gamma(1,0)=\Gamma(1)=1$ for the incomplete gamma function in (4.38), can be safely taken, yielding

$$\lim_{\Lambda^2\to\infty}K(x,y;\Lambda^2)=\frac{3ai}{2\pi^2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{(ip)^6}{(ip)(i\bar{p})}\,e^{ip\cdot(x-y)}=\frac{3ai}{2\pi^2}\,\frac{\partial^5}{\bar{\partial}}\,\delta^2(x-y),$$

with the numerical factor a given by the integral

$$a=\int_0^{1/3}\mathrm{d}y\int_{2y}^{(1+y)/2}\mathrm{d}x\frac{[(1-x)(x-y)y]^3}{[x(1-x)+y(x-y)]^6}.$$

The further change of variables $\{u=1/x,\ v=y/x\}$, followed by the subsequent change $u\to(u-1)/v(1-v)$, brings the above integral to a simpler form, giving finally the result

$$a=\int_0^{1/2}\mathrm{d}v\, v(1-v)\int_{1/v}^{\infty}\mathrm{d}u\frac{u^3}{(1+u)^6}=\frac{1}{720}.$$

All in all, taking the above results into account, the following regularized expression of $\Gamma(v^*,v^*)$ (4.35), in the limit $\Lambda^2\to\infty$, comes out

$$\Gamma(v^*,v^*)=\frac{\hbar^2d^2}{1440\pi^2}\int\mathrm{d}^2x\,v^*\,\frac{\partial^5}{\bar{\partial}}\,v^*,\tag{4.39}$$

whereas further calculation of its BRST variation, by using the transformation

$$\delta v^*=(v^*,S)=-\bar{\partial}u+O(\Phi^*),$$

finally yields, after nonlocal radiative corrections induced by the one-loop anomaly are separated off, the universal two-loop anomaly

$$\mathcal{A}_2(v^*,u)=\frac{id^2}{720\pi^2}\int\mathrm{d}^2x\,u\,\partial^5\,v^*,\tag{4.40}$$

to which expression (4.33) should be added when considering the one-loop counterterm M_1 (4.25) included in the theory.

Expressions (4.39) for (part of) the two-loop effective action and (4.40) for the universal two-loop anomaly, are in agreement –apart from numerical factors which can be traced back to the diferent form of the actions used in the computations– with earlier results in the literature [20, 21, 22, 23, 24]. This fact indicates thus that, at least for this case, the nonlocally regularized FA formalism is able to deal with two-loop corrections and reproduce the correct numerical coefficients for the anomalies.

5 Conclusions and Perspectives

The aim of this paper has been to extend to general gauge theories the nonlocal regularization method of ref. [1, 2, 3], by reformulating it according to the ideas of the antibracket-antifield formalism. As a result of this procedure, a nonlocal regularized form S_Λ (3.7) of the proper solution of the classical master equation S (1.3) comes out, which encodes all the information about the structure of the regulated BRST symmetry. Further extension of the basic ideas of nonlocal regularization at quantum level provides a systematic way to construct the nonlocal regularized version W_Λ (3.17) of the quantum action W (1.5). In the end, its use in the basic path integral leads to a fully regularized version of the FA formalism, which is able to treat higher order loop corrections and offers a suitable framework for the analysis of the BRST Ward identity (3.19) and of the BRST anomaly (3.28) proposed by FA through the quantum master equation.

From the analysis of this fundamental equation, based in the study of the action of the operator Δ (1.8) on the regularized quantum action W_Λ (3.17), and supplemented with explicit calculations performed in the example of chiral W_3 gravity, one of the main conclusions of this paper arises, namely, the incompleteness of the quantum master equation and its ability to describe only the one-loop corrections generated by the counterterms M_k, $k < p$, to the p-loop anomalies. In other words, the present form of the quantum master equation seems to be the "one-loop form" of a general expression, presumably defined in terms of suitable quantum generalizations of the operator Δ (1.8), $\Delta_q = \Delta + \hbar\Delta_1 + \ldots$, and of the antibracket (1.2) involved in its definition.

The incompleteness of the present form of the quantum master equation, on the other hand, can be traced back to the naive character of the FA derivations in which this quantity appears as obstruction. Indeed, apart from threatening the fulfillment of the BRST Ward identity, the quantum master equation also controls, for instance, the theory's (in)dependence of the gauge choice [4], as the following equation shows

$$Z_{\Psi+\delta\Psi} = Z_\Psi + \int \mathcal{D}\Phi \exp\left[\frac{i}{\hbar} W_\Sigma(\Phi)\right] \left[\Delta W + \frac{i}{2\hbar}(W,W)\right]_\Sigma \cdot \delta\Psi, \tag{5.1}$$

where the subindex Σ stands for the restricition to the so-called gauge fixing surface, defined in terms of a suitable gauge-fixing fermion Ψ as $\Sigma = \{\Phi^* = \frac{\partial\Psi}{\partial\Phi}\}$, and with Z_Ψ the value of the generating functional (1.4) in this surface when $J = 0$. This equation is derived by first varying infinitesimaly the gauge fermion Ψ in Z_Ψ to $\Psi + \delta\Psi$, and evaluating afterwards the resulting expression by considering the invariance of Z_Ψ under the change of variables $\Phi \to \Phi + \delta\Phi$, with $\delta\Phi$ given by

$$\delta\Phi^A = (\Phi^A, W)_\Sigma \cdot \delta\Psi. \tag{5.2}$$

In this way, the term ΔW in (5.1) originates in the noninvariance of the functional measure under (5.2), while (W,W) comes from the transformation of the integrand.

However, the treatment of field redefinitions at functional level in this way, namely, by merely changing integration variables in functional integrals, has been known to be incorrect for a long time. Indeed, in ref. [26], nonlinear point canonical transformations were analyzed at the quantum mechanical level, using a discretized version of the path integral, and the original and naively transformed theory were found to differ by terms of order $\hbar^2$. More recently, this question has been analyzed as well at the field theoretical level in [27] on the same grounds, with the result that, upon making nonlinear field redefinitions, extra terms of order $\hbar^2$ and higher generally appear together with the ones expected to come from the naive change of variables.

In view of these results and taking into account the nonlinear character of the BRST transformation (5.2) in general, it is natural to expect that the missing $O(\hbar^2)$ terms in the present form

of the quantum master equation, the ones which shall characterize higher order loop anomalies, should come precisely from the extra $O(\hbar^2)$ pieces generated upon performing the change of variables (5.2), i.e., upon making in the proper way changes of variables in the path integral. Implementation of these ideas in the nonlocally regulated FA formalism, in order to translate to the continuum the lattice expressions obtained in [27], would then be useful in order to explicitly determine the form of the higher order terms involved in the complete form of the quantum master equation and, as a consequence, of the higher order loop anomalies, without relying on diagramatic calculations. In the end, such expressions would also possibly allow an algebraic characterization of higher order loop anomalies, in much the same way as one-loop anomalies are characterized as being non-trivial BRST cocycles at ghost number one. Work towards this algebraic approach to characterize anomalies by means of a "complete" quantum master equation is already in progress.

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A Appendix: Functional Traces

This appendix is devoted to the computation of the general form of the functional traces involved in the computations performed in section 4. The first of these traces appears in the evaluation of the divergent term of Ω_0 (4.10)

$$\mathcal{T}(F,n) = \mathrm{Tr}\left[\varepsilon^2 (F(\Phi,\Phi^*)\,\partial)_n\right],$$

with the operators ε^2 and $(F(\Phi,\Phi^*)\,\partial)_n$ defined by (4.4) and (4.7), respectively. Introducing as usual momentum space eigenfunctions to saturate the sum, the above trace can be written as

$$\begin{aligned}\mathcal{T}(F,n) &= \int \mathrm{d}^2x \int \frac{\mathrm{d}^2k}{(2\pi)^2}\, e^{-ik\cdot x} \left\{[F\partial + n(\partial F)] \exp\left(\frac{\partial\bar{\partial}}{\Lambda^2}\right)\right\} e^{ik\cdot x} \\ &= \int \mathrm{d}^2x \int \frac{\mathrm{d}^2k}{(2\pi)^2}\, [ik\,F + n(\partial F)] \exp\left(\frac{-k\bar{k}}{\Lambda^2}\right).\end{aligned}$$

Further rescaling of the momentum k by Λ and Wick rotation to euclidian space, in order to obtain the usual gaussian damping factors, finally yields

$$\mathcal{T}(F,n) = -\Lambda^2 \,\frac{i\,n}{2\pi} \int \mathrm{d}^2x\,(\partial F) = 0, \tag{A.1}$$

as a consequence of the integrals

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \exp\left(-k\bar{k}\right) = \frac{-i}{2\pi}, \qquad \int \frac{\mathrm{d}^2k}{(2\pi)^2}\,(k \overset{n}{\cdots} k) \exp\left(-k\bar{k}\right) = 0, \qquad \forall\, n > 0, \tag{A.2}$$

and the fact that the integrand in (A.1) is a total derivative.

The second relevant integral, useful for the calculation of finite contributions to Ω_0 (4.10) and Ω_1 (4.28), is given by the general expression (4.17)

$$\mathcal{T}(F,G;n,m) = \lim_{\Lambda^2\to\infty} \mathcal{T}(F,G;n,m;\Lambda^2) = \lim_{\Lambda^2\to\infty} \mathrm{Tr}\left[\varepsilon^2\, F\,\partial^n\,\mathcal{O}\partial\, G\,\partial^m\right], \quad \text{for} \quad n,m \geq 0.$$

Upon use of momentum space eigenfunctions, as in the previous computation, and further introduction of the momentum representation $G(x) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} G(p) e^{ip\cdot x}$ yields

$$\mathcal{T}(F,G;n,m;\Lambda^2) = \int \mathrm{d}^2x\, F(x) \left\{ \int \frac{\mathrm{d}^2k}{(2\pi)^2} \int \frac{\mathrm{d}^2p}{(2\pi)^2} \right.$$
$$\left. \left[\int_0^1 \frac{\mathrm{d}t}{\Lambda^2} \exp\left(-t \frac{(k+p)(\bar{k}+\bar{p})}{\Lambda^2} - \frac{k\bar{k}}{\Lambda^2} \right) \right] (ik)^m \, (ip+ik)^{n+1} G(p) e^{ip\cdot x} \right\}, \qquad \text{(A.3)}$$

The integrals over k and p can now be decoupled by performing the standard change of variables

$$k^\mu \;\to\; \left(\frac{\Lambda}{\sqrt{1+t}}\, k^\mu - \frac{t}{1+t}\, p^\mu \right).$$

The factor $[(ik)^m (ip+ik)^{n+1}]$ in the integrand of (A.3) produces then a polynomial in k, of which only the k-independent term survives, as a consequence of (A.2). The result of these manipulations is thus

$$\mathcal{T}(F,G;n,m;\Lambda^2) =$$
$$\frac{-i}{2\pi} \int \mathrm{d}^2x\, F(x) \left\{ \int \frac{\mathrm{d}^2p}{(2\pi)^2} \left[\int_0^1 \mathrm{d}t\, \frac{(-1)^m\, t^m}{(1+t)^{n+m+2}} \exp\left(-\frac{t}{1+t} \frac{p\bar{p}}{\Lambda^2} \right) \right] (ip)^{n+m+1} G(p) e^{ip\cdot x} \right\} =$$
$$\frac{-i}{2\pi} \int \mathrm{d}^2x\, F\, \bar{\mathcal{O}}(n,m;\Lambda^2)\, \partial^{n+m+1} G,$$

where the factor $-i/2\pi$ comes from integration over the internal momentum k, and where the operator $\bar{\mathcal{O}}(n,m;\Lambda^2)$ is defined by

$$\bar{\mathcal{O}}(n,m;\Lambda^2) = \int_0^1 \mathrm{d}t\, \frac{(-1)^m\, t^m}{(1+t)^{n+m+2}} \exp\left(\frac{t}{1+t} \frac{\partial\bar{\partial}}{\Lambda^2} \right).$$

The limit $\Lambda^2 \to \infty$ of the above operator is well defined, yielding

$$\begin{aligned} \bar{\mathcal{O}}(n,m) &= \lim_{\Lambda^2\to\infty} \bar{\mathcal{O}}(n,m;\Lambda^2) = \int_0^1 \mathrm{d}t\, \frac{(-1)^m\, t^m}{(1+t)^{n+m+2}} \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{n+m+1-k} \left(1 - \frac{1}{2^{n+m+1-k}} \right), \end{aligned} \qquad \text{(A.4)}$$

and leading to the final result (4.18)

$$\mathcal{T}(F,G;n,m) = \frac{-i}{2\pi}\, \bar{\mathcal{O}}(n,m) \int \mathrm{d}^2x\, F\, \partial^{n+m+1} G, \quad \text{for} \quad n,m \geq 0,$$

with $\bar{\mathcal{O}}(n,m)$ given by (A.4).

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